SMOOTHNESS OF EQUIVARIANT DERIVED CATEGORIES

VALERY A. LUNTS AND OLAF M. SCHNÜRER

Abstract. We introduce the notion of (homological) G-smoothness for a complex G-variety X, where G is a connected affine algebraic group. This is based on the notion of smoothness for dg algebras and uses a suitable enhancement of the G-equivariant derived category of X. If there are only finitely many G-orbits and all stabilizers are connected, we show that X is G-smooth if and only if all orbits $\mathcal{O}$ satisfy $H^*(\mathcal{O};\mathbb{R}) = \mathbb{R}$. On the way we prove several results concerning smoothness of dg categories over a graded commutative dg ring.

Contents

1. Introduction

We introduce the notion of (homological) G-smoothness of a complex G-variety X, where G is a connected complex affine algebraic group. The idea is to describe the G-equivariant morphism $X \to \mathrm{pt}$ in terms of dg algebras and to use the notion of smoothness defined for such algebras. Under suitable conditions we give sufficient and necessary conditions for G-smoothness of X. This is based on results on smoothness for dg K-algebras (or categories) over a graded commutative dg ring K that are of independent interest.

We first explain the definition and results concerning G-smoothness. We assume in the following that G acts on X with finitely many orbits and that all stabilizer subgroups are connected. We work with the G-equivariant bounded constructible derived category $D^b_{G,c}(X)$ of sheaves of real vector spaces on X (see [BL94]). Using a suitable enhancement we find a dg $H_G(\mathrm{pt})$-algebra A such that the perfect derived category $\mathrm{per}(A)$ of dg A-modules is equivalent to $D^b_{G,c}(X)$. The structure morphism $H_G(\mathrm{pt}) \to A$ may be thought

of as an analog of the G-morphism $X \to \mathrm{pt}$ (recall that $D^b_{G,c}(\mathrm{pt})$ and $\mathrm{per}(H_G(\mathrm{pt}))$ are equivalent). Slightly generalizing the standard definition we say that A is $H_G(\mathrm{pt})$-smooth if the diagonal bimodule A is in $\mathrm{per}(A \otimes^L_{H_G(\mathrm{pt})} A^{\mathrm{op}})$. Then we define X to be G-smooth if A is $H_G(\mathrm{pt})$-smooth. Our first main result shows that G-smoothness can be tested on the orbits.

Theorem 1.1 (see Theorem 4.9)**.** *Under the above conditions, X is G-smooth if and only if all G-orbits in X are G-smooth.*

Hence we need to understand when an orbit is G-smooth. Our second main result gives a criterion answering this question.

Theorem 1.2 (see Theorem 4.8 and references there)**.** *Let $\mathcal{O} = G/H$ where G is as above and $H \subset G$ is a closed connected subgroup. Then the following conditions are equivalent:*

- *(a) $\mathcal{O}$ is G-smooth.*
- *(b) $H_G(pt) \to H_G(\mathcal{O}) = H_H(pt)$ is an isomorphism.*
- *(c) $H_H(pt)$ is a smooth dg $H_G(pt)$-algebra.*
- *(d) Any maximal compact subgroup of H is a maximal compact subgroup of G.*
- *(e) $H^*(\mathcal{O};\mathbb{R}) = \mathbb{R}$.*
- *(f) $\mathcal{O} \cong \mathbb{C}^n$ as complex varieties for some $n \in \mathbb{N}$.*

There are some more equivalent conditions given in Theorem 4.8 which are actually needed for the proof of Theorem 1.1.

For example, if G is reductive and $B \subsetneq G$ is a Borel subgroup, then the flag variety G/B is B-smooth but not G-smooth.

We include some details how the above dg $H_G(\mathrm{pt})$-algebra A is defined. We fix a suitable universal G-principal fiber bundle $EG \to BG$. Among other things, this means that there is a (version of the) de Rham sheaf Ω_{BG} of dg algebras on BG computing $H_G(\mathrm{pt})$. Let $c : X_G := EG \times_G X \to BG$ be the obvious map. We identify $D^b_{G,c}(X)$ with a full subcategory of the derived category of dg $c^*(\Omega_{BG})$-modules (= sheaves of dg modules over the sheaf $c^*(\Omega_{BG})$ of dg algebras) on X_G. Using an injective model structure we find a dg enhancement $\mathcal{E}$ of $D^b_{G,c}(X)$ that consists of h-injective dg $c^*(\Omega_{BG})$-modules. Let $E \in \mathcal{E}$ be an object that is a classical generator of $D^b_{G,c}(X)$. Then in the obvious way $A := \mathcal{E}(E,E)$ is a dg $\Gamma(\Omega_{BG})$-algebra. By composing the structure morphism with a quasi-isomorphism $H_G(\mathrm{pt}) \to \Gamma(\Omega_{BG})$ we obtain a dg $H_G(\mathrm{pt})$-algebra A that we can use for the definition of G-smoothness of X, as explained above.

The proofs of the above Theorems rely on some results on dg K-categories, where K is a graded commutative dg ring, that we present now. A dg K-category is by definition a category enriched in the (abelian symmetric) monoidal category $C(K)$ of dg K-modules. For

example, a dg K-category with one object is a dg K-algebra. The obvious generalization of a result of G. Tabuada [Tab05] shows that the category of small dg K-categories carries a cofibrantly generated model structure whose weak equivalences are the quasi-equivalences. In particular, any dg K-category (or algebra) $\mathcal{A}$ has a cofibrant replacement $Q(\mathcal{A}) \to \mathcal{A}$ which allows us to define $\mathcal{A} \otimes^L_K \mathcal{A} = Q(\mathcal{A}) \otimes_K Q(\mathcal{A})$. We say that $\mathcal{A}$ is K-smooth if the diagonal bimodule $\mathcal{A}$ is in $\mathrm{per}(\mathcal{A} \otimes^L_K \mathcal{A})$. Here, if $\mathcal{B}$ is a dg K-category, $\mathrm{per}(\mathcal{B})$ is the perfect derived category of dg $\mathcal{B}$-modules, which can also be characterized as the full subcategory of the derived category of dg $\mathcal{B}$-modules consisting of compact objects. Instead of $Q(\mathcal{A}) \to \mathcal{A}$ we can take any trivial fibration $\mathcal{A}' \to \mathcal{A}$ with $\mathcal{A}'$ K-h-flat for testing K-smoothness of $\mathcal{A}$. Then $\mathcal{A}$ is K-smooth if and only if the diagonal bimodule $\mathcal{A}$ is in $\mathrm{per}(\mathcal{A}' \otimes_K \mathcal{A}')$.

We generalize and strengthen two results of [Lun10]: Theorem 3.17 says that K-smoothness is invariant under dg Morita equivalence, i. e. if the derived categories of two dg K-categories $\mathcal{A}$ and $\mathcal{B}$ are connected by a zig-zag of tensor equivalences, then $\mathcal{A}$ is K-smooth if and only if $\mathcal{B}$ is K-smooth. Theorem 3.24 shows the following: If A and B are dg K-algebras (or categories) and N is a dg $B \otimes_K A^{\mathrm{op}}$-module, then the dg K-algebra (or category) $\left[\begin{smallmatrix} B & 0 \\ N & A \end{smallmatrix}\right]$ is K-smooth if and only if A and B are K-smooth and N is an object of $\mathrm{per}(B \otimes^L_K A^{\mathrm{op}})$. This result is the key to the proof of Theorem 1.1: A decomposition of X into an open G-orbit and its closed complement provides a dg $H_G(\mathrm{pt})$-algebra of this form.

For the proof of Theorem 1.2 we need two more results. The first one is Theorem 3.30 and explains how K-smoothness behaves with respect to a base change: Let $K \to K'$ be a morphism of dg rings. If a dg K-category $\mathcal{A}$ is K-smooth, then $\mathcal{A} \otimes^L_K K' := Q(\mathcal{A}) \otimes_K K'$ is K'-smooth. Moreover, the converse is true if $K \to K'$ is a quasi-isomorphism. The second result is the slightly technical criterion for K-smoothness given in Proposition 3.43. It is based on an easier criterion, stated in Proposition 3.40: Let A be a dg algebra over a field k that satisfies $H^i(A) = 0$ for $i < 0$, $H^0(A) = \mathsf{k}$ and $H^i(A) = 0$ for $i \gg 0$. Then A is k-smooth if and only if $H(A) = \mathsf{k}$.

Acknowledgment. We would like to thank Hanspeter Kraft and Bertrand Toën for useful correspondence. For helpful comments, mainly on model categories, we are grateful to Tobias Dyckerhoff and Steffen Sagave.

The first author was partially supported by the NSF grant 0901301.

The second author is grateful to the first author and his family for their hospitality during his visit to Bloomington. He also thanks Indiana University and in particular the people in the math department for their hospitality. He is thankful to the Collaborative Research

Center SFB Transregio 45 of the German Science foundation for support. He was partially supported by the priority program SPP 1388 of the German Science foundation.

2. Differential graded K-categories

This Section generalizes in a straightforward manner well known results from dg (= differential graded) categories over a commutative ring k to dg categories over a graded commutative dg algebra K; for example we describe the projective model structure on the category of modules over such a dg K-category, and we equip the category dgcat_K of small dg K-categories with a model structure (following G. Tabuada [Tab05]). The reader who is familiar with the usual theory will find no surprises and is advised to pass directly to Section 3.

Let k be a commutative (associative unital) ring and K a graded commutative dg (= differential ($\mathbb{Z}$-)graded) (k-)algebra, i. e. $K = \bigoplus_{p\in\mathbb{Z}} K^p$ is a graded (associative unital) k-algebra (the structure morphism $\mathsf{k} \to K$ lands in K^0 and in the center of K) endowed with a k-linear differential $d = (d^p : K^p \to K^{p+1})_{p\in\mathbb{Z}}$ of degree one such that $d(kl) = d(k)l + (-1)^{|k|}kd(l)$ for all elements $k,\ l \in K$ with k of degree $|k|$ (here and in the following we use the convention that elements are assumed to be homogeneous if their degree appears in a formula); the assumption that K is graded commutative means that $kl = (-1)^{|k||l|}lk$ for all $k,\ l \in K$. For example, K could be k viewed as a dg algebra concentrated in degree zero. We fix k and K for the rest of this article.

2.1. **Dg K-categories.** Let $C(K)$ be the abelian k-linear category of (right) dg K-modules (morphisms are K-linear, preserve the degree and commute with the respective differentials). Given dg K-modules $M,\ N,$ we can view N as a left dg K-module via $k.n := (-1)^{|k||n|}nk$ and obtain the tensor product $M \otimes_K N$ which is again an a dg K-module. In fact $C(K)$ becomes a symmetric monoidal category in the obvious way, which is moreover closed: Given any $M \in C(K)$, the functor $(? \otimes_K M)$ has an obvious right adjoint denoted $\underline{\mathrm{Hom}}(M, ?)$, i. e. $(C(K))(N \otimes_K M, P) = (C(K))(N, \underline{\mathrm{Hom}}(M, P))$ naturally in N and P.

This enables us to speak about dg K-categories (:= $C(K)$-(enriched)categories), dg K-functors (:= $C(K)$-functors), and dg K-natural transformations (:= $C(K)$-natural transformations), see [Kel05].

To an arbitrary dg K-category $\mathcal{A}$ we can associate two k-linear categories, namely the category $Z^0(\mathcal{A})$ and the homotopy category $[\mathcal{A}]$. They have the same objects as $\mathcal{A}$, but their morphisms spaces are given by the cocycles $(Z^0(\mathcal{A}))(A, A') = Z^0(\mathcal{A}(A, A))$ of degree zero and by the cohomology classes $[\mathcal{A}](A, A') = H^0(\mathcal{A}(A, A'))$ of degree zero.

An example is the dg K-category $\mathcal{M}od(K)$ of dg K-modules. It has the same objects as $C(K)$, and its morphism spaces are given by $(\mathcal{M}od(K))(M,N) = \underline{\mathrm{Hom}}(M,N)$ where M, N are dg K-modules. Note that

$$(C(K))(M,N) = (C(K))(K, (\mathcal{M}od(K))(M,N)) = Z^0((\mathcal{M}od(K))(M,N)).$$

The first equality says that the underlying category of the dg K-category $\mathcal{M}od(K)$ is $C(K)$, and then the second equality says that $C(K) = Z^0(\mathcal{M}od(K))$.

2.2. **Module categories.** Let $\mathcal{A}$ be a small dg K-category. A (right) dg $\mathcal{A}$-module M is a dg K-functor $M : \mathcal{A}^{\mathrm{op}} \to \mathcal{M}od(K)$, where $\mathcal{A}^{\mathrm{op}}$ is the opposite dg K-category. More explicitly, such a functor is given by dg K-modules $M(A)$, for $A \in \mathcal{A}$, and morphisms

$$M(A) \otimes_K \mathcal{A}(A', A) \to M(A')$$

in $C(K)$, for A, $A' \in \mathcal{A}$, that make the obvious diagrams encoding unitality and associativity commutative. We denote the category of dg $\mathcal{A}$-modules whose morphisms are the dg K-natural transformations by $C(\mathcal{A})$. This is an abelian k-linear category having all small limits and colimits; we explain in Remark 2.1 below that it is essentially independent of K.

Again there is a dg K-category $\mathcal{M}od(\mathcal{A})$ whose underlying category is $C(\mathcal{A})$. Let M, N be dg $\mathcal{A}$-modules. Then $(\mathcal{M}od(\mathcal{A}))(M,N)$ is defined to be the dg K-module

$$\Big\{ (f(A)) \in \prod_{A \in \mathcal{A}} (\mathcal{M}od(K))(M(A), N(A)) \mid \; N(a)f(A') = f(A'')M(a) \text{ for all } a \in \mathcal{A}^{\mathrm{op}}(A', A'') \Big\}.$$

Similar as above we have

$$C(\mathcal{A})(M,N) = Z^0((\mathcal{M}od(\mathcal{A}))(M,N)).$$

We may consider K as a dg K-category consisting of one object whose endomorphisms are K. Then the definitions of $C(K)$ and $\mathcal{M}od(K)$ are consistent with their previous definitions.

Any object $A \in \mathcal{A}$ gives rise to the dg $\mathcal{A}$-module $\widehat{A} := \mathcal{A}(?, A)$ represented by A. If M is any dg $\mathcal{A}$-module, the map

$$(\mathcal{M}od(\mathcal{A}))(\widehat{A}, M) \xrightarrow{\sim} M(A), \quad f \mapsto (f(A))(\mathrm{id}_A), \tag{2.1}$$

is an isomorphism in $C(K)$, the Yoneda-isomorphism. Taking degree zero cocycles gives the isomorphism

$$(C(\mathcal{A}))(\widehat{A}, M) \xrightarrow{\sim} Z^0(M(A)).$$

The dg K-functor

$$\mathcal{A} \to \mathcal{M}od(\mathcal{A}), \quad A \mapsto \widehat{A} = \mathcal{A}(?, A),$$

is full and faithful by (2.1) and called the Yoneda embedding.

Whenever we work with module categories over a dg K-category in the following we implicitly assume that the given dg K-category is small.

2.3. **Homotopy categories and derived categories.** We have seen above that $Z^0(\mathcal{M}od(\mathcal{A})) = C(\mathcal{A})$. We define $\mathcal{H}(\mathcal{A}) := [\mathcal{M}od(\mathcal{A})]$ and call it the homotopy category of dg $\mathcal{A}$-modules. There is an obvious functor $C(\mathcal{A}) \to \mathcal{H}(\mathcal{A})$.

In the usual way (cf. e. g. [BL94, Ch. 10]) we equip $\mathcal{H}(\mathcal{A})$ with the structure of a triangulated category: One defines the translation or shift functor $[1]$ on $\mathcal{M}od(\mathcal{A})$ (and $C(\mathcal{A})$, $\mathcal{H}(\mathcal{A})$), and the cone $\operatorname{Cone}(f)$ of a morphism $f : M \to N$ in $C(\mathcal{A})$; this cone fits into an obvious diagram $M \xrightarrow{f} N \to \operatorname{Cone}(f) \to [1]M$ in $C(\mathcal{A})$ called a standard triangle. We define a (distinguished) triangle in $\mathcal{H}(\mathcal{A})$ to be a candidate triangle isomorphic to the image of a standard triangle. Then $\mathcal{H}(\mathcal{A})$ with the shift functor $[1]$ and this class of triangles is a triangulated category.

A morphism $f : M \to N$ in $C(\mathcal{A})$ (or $\mathcal{H}(\mathcal{A})$) induces in the obvious way a morphism $H(f) : H(M) \to H(N)$ on cohomology. We call f a quasi-isomorphism if $H(f)$ is an isomorphism.

A dg $\mathcal{A}$-module M is called acyclic if all $M(A)$ have vanishing cohomology, i. e. $H(M(A)) = 0$, for all $A \in \mathcal{A}^{\mathrm{op}}$. Then a morphism in $\mathcal{H}(\mathcal{A})$ is a quasi-isomorphism if and only if its cone is acyclic; here we mean by the cone of a morphism f in a triangulated category the third object in a triangle whose first morphism is f (it is well defined up to isomorphism).

The derived category $D(\mathcal{A})$ of dg $\mathcal{A}$-modules is defined to be the Verdier quotient of $\mathcal{H}(\mathcal{A})$ by the (thick) triangulated subcategory of all acyclic dg $\mathcal{A}$-modules. Note that $\mathcal{H}(\mathcal{A})$ and $D(\mathcal{A})$ have all small coproducts and products, and the functor $\mathcal{H}(\mathcal{A}) \to D(\mathcal{A})$ commutes with coproducts and products.

A dg $\mathcal{A}$-module P is called h-projective if all morphisms $P \to N$ in $C(\mathcal{A})$ with acyclic N are homotopic to zero, i. e. $(\mathcal{H}(\mathcal{A}))(P, N) = 0$. For example all $[i]\widehat{A}$ for $A \in \mathcal{A}$ and $i \in \mathbb{Z}$ are h-projective since

$$(\mathcal{H}(\mathcal{A}))([i]\widehat{A}, N) \xrightarrow{\sim} H^{-i}(N(A)). \tag{2.2}$$

by (2.1).

If P and M are dg $\mathcal{A}$-modules it is easy to see that the canonical morphism

$$(\mathcal{H}(\mathcal{A}))(P, M) \to (D(\mathcal{A}))(P, M) \tag{2.3}$$

is an isomorphism if P is h-projective.

We define $\operatorname{per}(\mathcal{A})$ to be the smallest strict full triangulated subcategory of $D(\mathcal{A})$ that contains all dg $\mathcal{A}$-modules $\widehat{A}$, for $A \in \mathcal{A}$, and is closed under summands. This category

has an alternative description. Let $D(\mathcal{A})^c$ be the full subcategory of $D(\mathcal{A})$ consisting of compact objects, i.e. objects M such that $(D(\mathcal{A}))(M,?)$ commutes with all coproducts. Let $\mathcal{E}$ be the set of all objects $[i]\widehat{A} \in D(\mathcal{A})$, for $A \in \mathcal{A}$ and $i \in \mathbb{Z}$. From (2.3) and (2.2) we deduce that $\mathcal{E}$ consists of compact objects, and moreover that $\mathcal{E}$ generates $D(\mathcal{A})$. The arguments of [Nee92] (cf. [BvdB03, Thm. 2.1.2]) show that

$$\operatorname{per}(\mathcal{A}) = D(\mathcal{A})^c \tag{2.4}$$

and that $D(\mathcal{A})$ is the smallest strict full triangulated subcategory of $D(\mathcal{A})$ that contains $\mathcal{E}$ and is closed with respect to the formation of arbitrary $D(\mathcal{A})$-coproducts, i.e. the localizing subcategory of $D(\mathcal{A})$ generated by $\mathcal{E}$ is all of $D(\mathcal{A})$.

Remark 2.1. *Let $Z \to K$ be a morphism of graded commutative dg algebras (for example the structure morphism $\mathsf{k} \to K$). Let $\mathcal{A}$ be a dg K-category. We define $\operatorname{res}^K_Z(\mathcal{A})$ to be the dg Z-category which is obtained from $\mathcal{A}$ by the obvious restriction along $Z \to K$.*

Then one checks that the obvious restriction functor

$$\operatorname{res}^{\mathcal{A}}_{\operatorname{res}^K_Z \mathcal{A}} : C(\mathcal{A}) \xrightarrow{\sim} C(\operatorname{res}^K_Z(\mathcal{A})) \tag{2.5}$$

is an isomorphism of k-linear categories. This is just an elaborate version of the following fact. If $R' \to R$ is a morphism of rings, and A is an R-algebra, then the module categories of A as an R-algebra and as an R'-algebra coincide (and only depend on the ring underlying A).

The above isomorphism (2.5) in fact comes from an isomorphism $\operatorname{res}^K_Z(\mathcal{M}od(\mathcal{A})) \xrightarrow{\sim} \mathcal{M}od(\operatorname{res}^K_Z \mathcal{A})$ of dg Z-categories. Similarly, we have isomorphisms $\mathcal{H}(\mathcal{A}) \xrightarrow{\sim} \mathcal{H}(\operatorname{res}^K_Z(\mathcal{A}))$ and $D(\mathcal{A}) \xrightarrow{\sim} D(\operatorname{res}^K_Z(\mathcal{A}))$ of k-linear categories.

2.4. **Projective model structure for dg $\mathcal{A}$-modules.** We refer to [Hov99a] (and [Lur09, App.]) for the language of model categories. However we do not assume that functorial factorizations are part of a model structure. Since all model categories we consider will be cofibrantly generated we can fix such factorizations whenever convenient.

We use the following terminology. If (P) is a property of objects in a model category, we define a **(P) resolution** (of an object X) to be a trivial fibration whose domain has property (P) (and whose codomain is X). It follows for example from the definition of a model category that any object has a cofibrant resolution.

Let $\mathcal{A}$ be a dg K-category. For $A \in \mathcal{A}$ and $n \in \mathbb{Z}$ define dg $\mathcal{A}$-modules $S_{n,A} := [n]\widehat{A}$ and $D_{n,A} := \operatorname{Cone}(\operatorname{id}_{S_{n,A}})$. There are obvious morphisms $\iota_{n,A} : S_{n,A} \to D_{n,A}$ in $C(\mathcal{A})$.

Define the following sets of morphisms in $C(\mathcal{A})$:

$$I := \{S_{n,A} \xrightarrow{\iota_{n,A}} D_{n,A} \mid A \in \mathcal{A}, n \in \mathbb{Z}\},$$
$$J := \{0 \to D_{n,A} \mid A \in \mathcal{A}, n \in \mathbb{Z}\}.$$

Theorem 2.2. *Let $\mathcal{A}$ be a (small) dg K-category. The category $C(\mathcal{A})$ can be equipped with the structure of a cofibrantly generated model category whose weak equivalences are the quasi-isomorphisms and whose fibrations are the epimorphisms. One can take I as the set of generating cofibrations and J as the set of generating trivial cofibrations.*

We call this model structure on $C(\mathcal{A})$ the ***projective*** *model structure.*

Remark 2.3. *Theorem 2.2 seems to be well known, at least for $K = \mathsf{k}$ (cf.* [Kel06, Thm. 3.2] *or* [Toë11, 3.2]*) and in the dg algebra case* [Fre09, 11.2.6]*. Presumably one can deduce its existence also from* [Lur09, App. 3] *and even see that it is a $C(K)$-model structure. Our approach is elementary and essentially follows* [Hov99a, Section 2.3]*.*

Proof. Let $\mathcal{W}$ be the class of quasi-isomorphisms in $C(\mathcal{A})$. Adapting the method of [Hov99a, Section 2.3] (and using the notation explained there) one proves that J-inj consists precisely of epimorphisms, that $I\text{-inj} = \mathcal{W} \cap J\text{-inj}$, that projective objects of $C(\mathcal{A})$ are acyclic and that J-cof consists precisely of (split) monomorphisms with cokernel a projective object of $C(\mathcal{A})$, so in particular $J\text{-cof} \subset \mathcal{W}$. Then application of [Hov99a, Thm 2.1.19 and Lemma 2.1.10] shows the result. □

Note that any object of $C(\mathcal{A})$ is fibrant. Examples of cofibrant objects are the objects $\widehat{A}$ and their shifts (take the pushout of a map in I along the morphism to the zero object).

The proof of [Hov99a, 2.3.9], adapted[1] to our setting, shows that the cofibrations are precisely the monomorphisms with cofibrant cokernel. This fact and the trivial fact that cofibrations are closed under composition shows the following two Lemmata.

Lemma 2.4. *Let $f : M \to N$ be a morphism in $C(\mathcal{A})$ between cofibrant objects. Then the canonical morphism $N \to \operatorname{Cone}(f)$ is a cofibration, and in particular $\operatorname{Cone}(f)$ is cofibrant.*

Lemma 2.5. *Let $M' \hookrightarrow M \twoheadrightarrow M''$ be a short exact sequence in $C(\mathcal{A})$ and assume that M' and M'' are cofibrant. Then the inclusion $M' \hookrightarrow M$ is a cofibration and M is cofibrant. (In fact this short exact sequence is isomorphic to the standard short exact sequence $M' \hookrightarrow \operatorname{Cone}(\tau) \twoheadrightarrow M''$ for some morphism $\tau : [-1]M'' \to M'$ in $C(\mathcal{A})$.)*

[1] For this we need the following result whose proof is similar to the proof of [Hov99a, 2.3.6]: Let C be a cofibrant dg $\mathcal{A}$-module. Then given any epimorphism $p : M \to N$ in $C(\mathcal{A})$ and any morphism $f : C \to N$ in $\mathcal{M}od(\mathcal{A})$ of degree zero (i. e. a morphism of graded $\mathcal{A}$-modules), there is a morphism $\hat{f} : C \to M$ in $\mathcal{M}od(\mathcal{A})$ of degree zero such that $p\hat{f} = f$.

Lemma 2.6 (cf. [Hov99a, Lemma 2.3.8])**.** *Any cofibrant object of $C(\mathcal{A})$ is h-projective.*

Proof. Let $C \in C(\mathcal{A})$ be cofibrant. Let $f : C \to N$ be a morphism in $C(\mathcal{A})$ and assume that N is acyclic. Then the obvious epimorphism $p : [-1]\operatorname{Cone}(\mathrm{id}_N) \to N$ is a quasi-isomorphism and hence a trivial fibration. Since C is cofibrant there is a lift $h : C \to [-1]\operatorname{Cone}(\mathrm{id}_N)$ such that $ph = f$. This lift has the form $h = \left[\begin{smallmatrix} D \\ f \end{smallmatrix}\right]$ and commutes with the differential. This implies that $f = d_N(-D) + (-D)d_C$ and hence $f = 0$ in $\mathcal{H}(\mathcal{A})$. □

Denote by $C(\mathcal{A})_{cf}$ (resp. $\mathcal{H}(\mathcal{A})_{cf}$) the full subcategory of $C(\mathcal{A})$ (resp. $\mathcal{H}(\mathcal{A})$) consisting of cofibrant (and fibrant) objects. From Lemma 2.4 we see that $\mathcal{H}(\mathcal{A})_{cf}$ is a triangulated subcategory of $\mathcal{H}(\mathcal{A})$ (non-strict in general). Since any object of $C(\mathcal{A})$ has a cofibrant resolution, Lemma 2.6 and (2.3) immediately imply that the canonical triangulated functor

$$\mathcal{H}(\mathcal{A})_{cf} \xrightarrow{\sim} D(\mathcal{A}) \tag{2.6}$$

is an equivalence. We fix for any any dg $\mathcal{A}$-module M a cofibrant (and hence h-projective) resolution $p(M) \to M$ (we could even assume that $p : C(\mathcal{A}) \to C(\mathcal{A})_{cf}$ is a functor). Then $M \mapsto p(M)$ extends to a functor

$$p : D(\mathcal{A}) \to \mathcal{H}(\mathcal{A})_{cf} \tag{2.7}$$

which is quasi-inverse to (2.6). We will use p for (left-)deriving certain functors.

A dg $\mathcal{A}$-module F is **free** if it is isomorphic in $C(\mathcal{A})$ to a coproduct of shifts of objects $\widehat{A}$, where A varies in $\mathcal{A}$. A dg $\mathcal{A}$-module F is called **semi-free** (cf. [Dri04, 13.1, 14.8]) if it can be represented as the union of an increasing sequence of dg $\mathcal{A}$-submodules F_i (where $i \in \mathbb{N}$) such that $F_0 = 0$ and each quotient F_{i+1}/F_i is a free dg $\mathcal{A}$-module.

Lemma 2.7.

(a) *All semi-free dg $\mathcal{A}$-modules are cofibrant.*
(b) *Every cofibrant dg $\mathcal{A}$-module is a retract of a semi-free dg $\mathcal{A}$-module.*

Proof. (a) This follows from Lemma 2.5 since free dg $\mathcal{A}$-modules are cofibrant (and cofibrations are closed under transfinite compositions).

(b) Let $C \in C(\mathcal{A})$ be cofibrant. The obvious variation of [Dri04, Lemma 13.3, cf. 14.8] shows that there is a surjective quasi-isomorphism (= trivial fibration) $f : F \to C$ where F is a semi-free dg $\mathcal{A}$-module. Since C is cofibrant, id_C factors through the trivial fibration f, and hence C is a retract of F. □

2.5. **Tensor product and flatness.** Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be dg K-categories, and let $X = {}_{\mathcal{B}}X_{\mathcal{A}}$ be a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module and $Y = {}_{\mathcal{C}}X_{\mathcal{B}}$ a dg $\mathcal{B} \otimes_K \mathcal{C}^{\mathrm{op}}$-module. Their tensor product is the dg $\mathcal{A} \otimes_K \mathcal{C}^{\mathrm{op}}$-module $Y \otimes_{\mathcal{B}} X$ defined in the usual way (cf. [Kel94, 6.1] or [Dri04, 14.3]). This construction yields a dg K-functor

$$(? \otimes_{\mathcal{B}} ?) : \mathcal{M}od(\mathcal{B} \otimes \mathcal{C}^{\mathrm{op}}) \otimes_K \mathcal{M}od(\mathcal{A} \otimes \mathcal{B}^{\mathrm{op}}) \to \mathcal{M}od(\mathcal{A} \otimes \mathcal{C}^{\mathrm{op}}).$$

A dg $\mathcal{A}$-module M is **$\mathcal{A}$-homotopically-flat** (abbreviated **$\mathcal{A}$-h-flat**) (cf. [Dri04, 14.7]) if $M \otimes_{\mathcal{A}} X$ is acyclic whenever X is an acyclic dg $\mathcal{A}^{\mathrm{op}}$-module. An equivalent condition is that $(M \otimes_{\mathcal{A}} ?)$ preserves quasi-isomorphisms, i. e. whenever $f : X \to Y$ in $C(\mathcal{A}^{\mathrm{op}})$ is a quasi-isomorphism, then $\mathrm{id}_M \otimes_{\mathcal{A}} f : M \otimes_{\mathcal{A}} X \to M \otimes_{\mathcal{A}} Y$ is a quasi-isomorphism.

Note that this also defines the notion of K-h-flatness by considering K as a dg K-category with one object.

Note that a dg $\mathcal{A}^{\mathrm{op}}$-module M is $\mathcal{A}^{\mathrm{op}}$-h-flat if and only if $(? \otimes_{\mathcal{A}} M)$ preserves acyclics.

If M is a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module we say that it is $\mathcal{A}$-h-flat if each $M(?, B)$ (for $B \in \mathcal{B}^{\mathrm{op}}$) is $\mathcal{A}$-flat, i. e. if $(M \otimes_{\mathcal{A}} ?)$ maps acyclic dg $\mathcal{A}^{\mathrm{op}}$-modules to acyclic dg $\mathcal{B}^{\mathrm{op}}$-modules, and we define $\mathcal{B}^{\mathrm{op}}$-h-flatness similarly.

For example, all $\widehat{A}$ (for $A \in \mathcal{A}$) are $\mathcal{A}$-h-flat, since $\widehat{A} \otimes_{\mathcal{A}} X = X(A)$ canonically. More generally, the following is true.

Lemma 2.8. *Any cofibrant dg $\mathcal{A}$-module is $\mathcal{A}$-h-flat. In particular any cofibrant dg K-module is K-h-flat.*

Proof. A cofibrant dg $\mathcal{A}$-module is a summand of a semi-free dg $\mathcal{A}$-module (Lemma 2.7), and semi-free modules are obviously $\mathcal{A}$-h-flat (cf. [Dri04, 14.8]). □

Lemma 2.9. *Let $f : M \to N$ be a quasi-isomorphism in $C(K)$ and assume that M and N are K-h-flat. If L is any dg K-module, then $f \otimes_K \mathrm{id}_L : M \otimes_K L \to N \otimes_K L$ is a quasi-isomorphism.*

Proof. Let $r : L' \to L$ be a K-h-flat resolution of L (e. g. a cofibrant resolution, cf. Lemma 2.8). In the commutative diagram

$$\begin{tikzcd} M \otimes_K L' \arrow[r, "f \otimes_K \mathrm{id}_{L'}"] \arrow[d, "\mathrm{id}_M \otimes_K r"'] & N \otimes_K L' \arrow[d, "\mathrm{id}_N \otimes_K r"] \\ M \otimes_K L \arrow[r, "f \otimes_K \mathrm{id}_L"] & N \otimes_K L, \end{tikzcd}$$

$f \otimes_K \mathrm{id}_L$ is a quasi-isomorphism since the other three morphisms are quasi-isomorphisms. □

Proposition 2.10. *Let $\mathcal{B}$ be a dg K-category with cofibrant morphism spaces.*

(a) *If M is a cofibrant dg $\mathcal{B}$-module, then $M(B)$ is a cofibrant dg K-module, for any $B \in \mathcal{B}$, and is in particular K-h-flat.*

Let $\mathcal{A}$ be a dg K-category.

(b) *Let $\mathcal{R} = \mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$ or $\mathcal{R} = \mathcal{A} \otimes_K \mathcal{B}$. Let X be a cofibrant dg $\mathcal{R}$-module. Then $X^B := X(?, B)$ is a cofibrant dg $\mathcal{A}$-module, for every $B \in \mathcal{B}$. In particular any cofibrant dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module is $\mathcal{A}$-h-flat.*

Assume that $\mathcal{A}$ has cofibrant morphism spaces.

(c) *If X is a cofibrant dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module, then $X(A, B)$ is a cofibrant dg K-module, for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, and is in particular K-h-flat.*

Proof. The assertions concerning h-flatness follow from Lemma 2.8.

(b) implies (a): Take $\mathcal{A} = K$ and $\mathcal{R} = \mathcal{A} \otimes_K \mathcal{B} = \mathcal{B}$.

(b) and (a) imply (c): Obvious.

We need to prove (b). We only consider the case $\mathcal{R} = \mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$, the case $\mathcal{R} = \mathcal{A} \otimes_K \mathcal{B}$ follows by considering $\mathcal{B}^{\text{op}}$ instead of $\mathcal{B}$.

Any cofibrant dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module is a summand of a semi-free dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module by Lemma 2.7, and cofibrant dg $\mathcal{A}$-modules are stable under summands. Hence we can assume that X is a semi-free dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module. Then there is an increasing filtration $0 = F_0 \subset F_1 \subset \dots$ of X such that $X = \bigcup_{i \in \mathbb{N}} F_i$ and each quotient F_{i+1}/F_i is a free dg $\mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$-module.

Let $B \in \mathcal{B}$. Evaluation at B is exact and hence yields an increasing filtration $0 = F_0^B \subset F_1^B \subset \dots$ of X^B such that $X^B = \bigcup_{i \in \mathbb{N}} F_i^B$. An obvious induction using Lemma 2.5 and the fact that cofibrations are closed under transfinite compositions shows that it is sufficient to show that all F_{i+1}^B / F_i^B are cofibrant dg $\mathcal{A}$-modules.

Fix $i \in \mathbb{N}$. Then F_{i+1}/F_i is isomorphic to a coproduct of shifts of objects $\widehat{R}$ for $R \in \mathcal{R}$. Evaluation at B yields that F_{i+1}^B / F_i^B is isomorphic to a coproduct of shifts of objects $\widehat{R}^B$ for $R \in \mathcal{R}$.

Hence it is sufficient to show that $\widehat{(A_0, B_0)}^B$ is cofibrant for any $(A_0, B_0) \in \mathcal{A} \otimes_K \mathcal{B}^{\text{op}}$. Since $\mathcal{B}(B_0, B)$ is a cofibrant dg K-module, it is a direct summand of a semi-free dg K-module G. We can write G as the union/colimit of a sequence $0 = G_0 \subset G_1 \subset \dots$ of dg K-submodules such that all quotients G_{j+1}/G_j are free dg K-modules. Fix isomorphisms $G_{j+1}/G_j \cong \bigoplus_{l \in L^{(j+1)}} [n_l^{(j+1)}]K$ for suitable index sets $L^{(j+1)}$.

For $A \in \mathcal{A}$ we have

$$\widehat{(A_0, B_0)}^B (A) = (\mathcal{A} \otimes_K \mathcal{B}^{\text{op}})((A, B), (A_0, B_0)) = \mathcal{A}(A, A_0) \otimes_K \mathcal{B}(B_0, B)$$

which is a direct summand of $\mathcal{A}(A, A_0) \otimes_K G$. Hence $\widehat{(A_0, B_0)}^B$ is a direct summand of $\widehat{A_0} \otimes_K G = \mathcal{A}(?, A_0) \otimes_K G$ and it is enough to show that the latter is a cofibrant dg $\mathcal{A}$-module.

Since $G_j \subset G_{j+1}$ splits in graded K-modules, $\mathcal{A}(?, A_0) \otimes_K G$ is the union/colimit of its submodules

$$0 = \mathcal{A}(?, A_0) \otimes_K G_0 \subset \mathcal{A}(?, A_0) \otimes_K G_1 \subset \dots$$

and the subquotients are isomorphic to

$$\mathcal{A}(?, A_0) \otimes_K (G_{j+1}/G_j) \cong \mathcal{A}(?, A_0) \otimes_K \bigoplus_{l \in L^{(j+1)}} [n_l^{(j+1)}]K = \bigoplus_{l \in L^{(j+1)}} [n_l^{(j+1)}]\mathcal{A}(?, A_0);$$

these claims can be checked by plugging in $A \in \mathcal{A}$.

Since coproducts of shifts of $\mathcal{A}(?, A_0) = \widehat{A_0}$ are cofibrant, an obvious induction using Lemma 2.5 (and a (countable) transfinite composition) shows that $\widehat{A_0} \otimes_K G$ is a cofibrant dg $\mathcal{A}$-module. □

2.6. **Standard functors and constructions.** We discuss some standard constructions and refer the reader to [Kel94, Section 6] and [Kel06] for more details.

Let $\mathcal{A}$, $\mathcal{B}$ be dg K-categories, and let $X = {}_{\mathcal{B}}X_{\mathcal{A}}$ be an $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module.

2.6.1. *Hom and tensor.* This datum gives rise to a pair (T_X, H_X) of adjoint dg K-functors

$$\mathcal{M}od(\mathcal{B}) \underset{H_X}{\overset{T_X}{\rightleftarrows}} \mathcal{M}od(\mathcal{A}) \tag{2.8}$$

where $T_X := (? \otimes_{\mathcal{B}} X)$, and $H_X(M)$ is defined by

$$(H_X(M))(B) := (\mathcal{M}od(\mathcal{A}))(X(?, B), M) \tag{2.9}$$

for $B \in \mathcal{B}$ with the obvious action morphisms.

2.6.2. *Dual bimodule.* Following [Kel94, 6.2] we define the dg $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$-module $X^{\perp} = {}_{\mathcal{A}}(X^{\perp})_{\mathcal{B}}$ by

$$X^{\perp}(B, A) := (\mathcal{M}od(\mathcal{A}))(X(?, B), \widehat{A}) \tag{2.10}$$

for $(B, A) \in \mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$ with obvious action morphisms. Observe that there is a canonical transformation $\tau : T_{X^{\perp}} = (? \otimes_{\mathcal{A}} X^{\perp}) \to H_X$ of dg K-functors.

2.6.3. *Left derived tensor product.* We define the triangulated functor $LT_X := (? \otimes^L_{\mathcal{B}} X)$ to be the composition

$$D(\mathcal{B}) \xrightarrow[\sim]{p} \mathcal{H}(\mathcal{B})_{cf} \xrightarrow{T_X} \mathcal{H}(\mathcal{A}) \to D(\mathcal{A}), \tag{2.11}$$

where the first arrow is the equivalence (2.7). Note that LT_X preserves all (small) coproducts. We call any functor $D(\mathcal{B}) \to D(\mathcal{A})$ of this form a tensor functor. A tensor equivalence is a tensor functor that is an equivalence.

2.6.4. *Restriction and extension of scalars.* Let $F : \mathcal{B} \to \mathcal{A}$ be a dg K-functor of dg K-categories. Taking $X = \mathcal{A}$ in (2.8) defines the extension of scalars dg K-functor $F^* := \operatorname{prod}^{\mathcal{A}}_{\mathcal{B}} := T_{\mathcal{A}} = (? \otimes_{\mathcal{B}} \mathcal{A})$ and its right adjoint $H_{\mathcal{A}}$ which is canonically isomorphic to the obvious restriction of scalars dg K-functor $F_* := \operatorname{res}^{\mathcal{A}}_{\mathcal{B}}$. Obviously $\operatorname{res}^{\mathcal{A}}_{\mathcal{B}}$ preserves acyclics and descends to a triangulated functor

$$\operatorname{res}^{\mathcal{A}}_{\mathcal{B}} : D(\mathcal{A}) \to D(\mathcal{B}). \tag{2.12}$$

It has $L\operatorname{prod}^{\mathcal{A}}_{\mathcal{B}} := LT_{\mathcal{A}}$ as a left adjoint, and this functor preserves compact objects since its right adjoint commutes with coproducts. If F is a quasi-equivalence (as defined below, see (qe1) and (qe2)), then (2.12) is an equivalence and induces an equivalence

$$\operatorname{res}^{\mathcal{A}}_{\mathcal{B}} : \operatorname{per}(\mathcal{A}) \xrightarrow{\sim} \operatorname{per}(\mathcal{B}). \tag{2.13}$$

This essentially follows from the results explained around (2.4) and the fact that $L\operatorname{prod}^{\mathcal{A}}_{\mathcal{B}}$ commutes with coproducts and maps $\widehat{B}$ to (an object isomorphic to) $\widehat{F(B)}$.

2.7. **Model structure on the category of dg K-categories.** G. Tabuada defines in [Tab05] a model structure on the category of small dg categories. We discuss a small generalization of this result.

Note that any dg K-functor $F : \mathcal{A} \to \mathcal{B}$ induces a functor $[F] : [\mathcal{A}] \to [\mathcal{B}]$ on homotopy categories. A dg K-functor $F : \mathcal{A} \to \mathcal{B}$ is a **quasi-equivalence** if

(qe1) for all objects $a_1, a_2 \in \mathcal{A}$, the morphism $F : \mathcal{A}(a_1, a_2) \to \mathcal{B}(Fa_1, Fa_2)$ is a quasi-isomorphism, and

(qe2) the functor $[F] : [\mathcal{A}] \to [\mathcal{B}]$ is essentially surjective (i. e. surjective on isoclasses of objects).

If (qe1) holds, then (qe2) is equivalent to the condition that $[F] : [\mathcal{A}] \to [\mathcal{B}]$ is an equivalence.

Denote by dgcat_K the category of small dg K-categories: Its objects are small dg K-categories, and its morphisms are dg K-functors.

Let Fib be the class of all morphisms $G : \mathcal{X} \to \mathcal{Y}$ in dgcat_K that satisfy the following two conditions:

(Fib1) for all objects x_1, x_2 of $\mathcal{X}$, the morphism $\mathcal{X}(x_1, x_2) \to \mathcal{Y}(Gx_1, Gx_2)$ is surjective, and

(Fib2) for all objects x_1 in $\mathcal{X}$ and each isomorphism $v : Gx_1 \to y$ in $[\mathcal{Y}]$ there is (an object x_2 in $\mathcal{X}$) and an isomorphism $u : x_1 \to x_2$ in $[\mathcal{X}]$ such that $[G](u) = v$ (so $Gx_2 = y$).

Theorem 2.11 (cf. [Tab05])**.** *The category* dgcat_K *can be equipped with the structure of a cofibrantly generated model category whose weak equivalences are the quasi-equivalences and whose fibrations are* Fib.

In the following, whenever we use model-theoretic terminology, we assume that all dg K-categories involved are small.

Proof. The proof of [Tab05, Thm. 2.1] generalizes in the obvious way to this setting. □

Note that any object of dgcat_K is fibrant. Moreover, the proof describes the class trFib of trivial fibrations (= morphisms that are weak equivalences and fibrations) as follows. A morphism $G : \mathcal{X} \to \mathcal{Y}$ in dgcat_K is a trivial fibration if and only if it is surjective in the following two senses:

(trFib1) for all objects $x_1, x_2 \in \mathcal{X}$, the morphism $G : \mathcal{X}(x_1, x_2) \to \mathcal{Y}(Gx_1, Gx_2)$ is a surjective quasi-isomorphism, and

(trFib2) G induces a surjection from the set of objects of $\mathcal{X}$ onto the set of objects of $\mathcal{Y}$.

A dg K-category $\mathcal{F}$ is **discrete** if $\mathcal{F}(X, X) = K$ for all $X \in \mathcal{F}$ and $\mathcal{F}(X, Y) = 0$ for all $X, Y \in \mathcal{F}$ with $X \neq Y$. A dg K-category $\mathcal{F}$ is **semi-free** (cf. [Dri04, 13.4]) if it can be represented as the union of an increasing sequence of dg K-subcategories $\mathcal{F}_i$ (where $i \in \mathbb{N}$) such that $\mathcal{F}_0$ is a discrete dg K-category and each $\mathcal{F}_i$ (for $i > 0$) as a graded K-category (= category enriched over the symmetric monoidal category of graded K-modules) is freely generated over $\mathcal{F}_{i-1}$ by a family of homogeneous morphisms f_α whose differentials df_α are morphisms in $\mathcal{F}_{i-1}$ (in particular all $\mathcal{F}_i$ have the same objects).

Lemma 2.12.

(a) All semi-free dg K*-categories are cofibrant.*

(b) Every cofibrant dg K*-category is a retract of a semi-free dg* K*-category.*

Proof. (a) A semi-free dg K-category has the required lifting property with respect to trivial fibrations, as follows from (the obvious variation of) [Dri04, 13.6].

(b) Let $\mathcal{C}$ be a cofibrant dg K-category. By the dg K-version of [Dri04, Lemma 13.5] there exists a semi-free dg K-category $\mathcal{F}$ and a trivial fibration $\mathcal{F} \to \mathcal{C}$ (which can be assumed to be the identity on objects). Since $\mathcal{C}$ is cofibrant $\mathrm{id}_{\mathcal{C}}$ factors as $\mathcal{C} \to \mathcal{F} \to \mathcal{C}$. □

Definition 2.13. *A dg K-category $\mathcal{A}$ is called K* ***-h-flat*** *if all morphisms spaces $\mathcal{A}(A, A')$, for $A,\ A' \in \mathcal{A}$, are K-h-flat.*

Lemma 2.14. *Cofibrant dg K-categories have cofibrant morphism spaces. In particular, they are K-h-flat, by Lemma 2.8.*

In the proof we will use the following obvious criterion for semi-freeness (generalized from [Dri04, 13.1.(2)] to our setting). Let M be a dg K-module. Then M is semi-free if (and only if) the following condition is satisfied: M has a homogeneous K-basis B (as a graded K-module) with the following property: for a subset $S \subset B$ let $\delta(S)$ be the smallest subset $T \subset B$ such that $d_M(S)$ is contained in the K-linear span of T; then for every $b \in B$ there is an $n \in \mathbb{N}$ such that $\delta^n(\{b\}) = \emptyset$.

Proof. Obviously retracts of dg K-categories with cofibrant morphism spaces have cofibrant morphism spaces. By Lemma 2.12 it is therefore sufficient to show that any semi-free dg K-category $\mathcal{F}$ has cofibrant morphism spaces. We even show that all $\mathcal{F}(A, A')$ are semi-free dg K-modules (for $A,\ A' \in \mathcal{F}$).

Choose an exhausting increasing filtration $(\mathcal{F}_i)_{i \in \mathbb{N}}$ of $\mathcal{F}$ such that $\mathcal{F}_0$ is a discrete dg K-category and $\mathcal{F}_{i+1} = \mathcal{F}_i\langle P_i \rangle$ (as a graded K-category) where P_i is a set of homogeneous arrows (= morphisms) such that the morphism dp is a cocycle in $\mathcal{F}_i$ for each $p \in P_i$.

This implies that the underlying graded K-category of $\mathcal{F}$ is freely generated by the arrows $\bigcup_{i \in \mathbb{N}} P_i$. Hence the set $B(A, A')$ of all paths in these arrows starting at A and ending at A' is a homogeneous K-basis of $\mathcal{F}(A, A')$. We claim that this basis satisfies the above criterion ensuring semi-freeness.

For $i \in \mathbb{N}$ denote the set of all paths in the arrows $\bigcup_{s<i} P_s$ by B_i, and by $B_i(X, X')$ the subset of paths that start at X and end at X' (for $X,\ X' \in \mathcal{F}_i$), so $B_i = \coprod_{X, X' \in \mathcal{F}_j} B_j(X, X')$. Then $B_i(X, X')$ is is a homogeneous K-basis of $\mathcal{F}_i(X, X')$. We fix i and can assume by induction that $B_i(X, X')$ satisfies the above criterion for all $X, X' \in \mathcal{F}$. We need to show that $B_{i+1}(A, A')$ satisfies this criterion.

Let $j \in \mathbb{N}$. Note that B_j is stable by composition (= concatenation of paths). If S and T are subsets of B_j, then obviously $\delta(S \cup T) = \delta(S) \cup \delta(T)$. Moreover the Leibniz rule $d(st) = d(s)t \pm sd(t)$ shows that $\delta(ST) \subset \delta(S)T \cup S\delta(T)$, and hence $\delta^n(ST) \subset \bigcup_{i=0}^n \delta^{n-i}(S)\delta^i(T)$.

For any $(a : X \to Y) \in P_i$ we know that $d(a) \subset \mathcal{F}_i(X, Y)$, hence $\delta(a) \subset T$ for some finite subset of B_i. By induction we know that $\delta^n(T) = \emptyset$ for n big enough and hence $\delta^{n+1}(\{a\}) = \emptyset$.

Any element b of B_{i+1} is a finite product of elements a (in B_i or in P_i) for which we already know that $\delta^n(\{a\}) = \emptyset$ for $n \gg 0$. Induction over the number of factors and

the above rule then show that $\delta^n(\{b\}) = \emptyset$ for $n \gg 0$. This proves that all $\mathcal{F}(A, A')$ are semi-free dg K-modules. $\square$

Lemma 2.15. *Let $G : \mathcal{R} \to \mathcal{S}$ be a quasi-equivalence of dg K-categories and let $\mathcal{F}$ be a K-h-flat dg K-category (e. g. a cofibrant dg K-category, cf. Lemma 2.14). Then $G \otimes_K \mathrm{id}_{\mathcal{F}} : \mathcal{R} \otimes_K \mathcal{F} \to \mathcal{S} \otimes_K \mathcal{F}$ is a quasi-equivalence.*

Proof. This is obvious. $\square$

3. Smoothness of dg K-categories

We generalize results of [Lun10, Section 3]. Recall that a K-h-flat resolution in dgcat_K is a trivial fibration $\mathcal{A}' \to \mathcal{A}$ such that $\mathcal{A}'$ is K-h-flat, and that a cofibrant resolution is a trivial fibration $\widetilde{\mathcal{A}} \to \mathcal{A}$ such that $\widetilde{\mathcal{A}}$ is cofibrant. Any object of dgcat_K has a cofibrant resolution, and cofibrant resolutions are K-h-flat by Lemma 2.14.

Lemma 3.1. *Let $f : \widetilde{\mathcal{A}} \to \mathcal{A}$ be a quasi-equivalence with cofibrant $\widetilde{\mathcal{A}}$ and $g : \mathcal{A}' \to \mathcal{A}$ a trivial fibration. Then there is a quasi-equivalence $h : \widetilde{\mathcal{A}} \to \mathcal{A}'$ such that $f = gh$.*

$$\begin{tikzcd} & \mathcal{A}' \arrow[d, "g"] \\ \widetilde{\mathcal{A}} \arrow[ur, dotted, "\exists h"] \arrow[r, "f"] & \mathcal{A} \end{tikzcd}$$

In particular, such a lift h exists if f is a cofibrant resolution and g is a K-h-flat resolution.

Proof. This follows directly from the definitions of a model category. $\square$

If N is a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module and $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{B}' \to \mathcal{B}$ are morphisms of dg K-categories, we denote the restriction of N along $\mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}} \to \mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$ by ${}_{\mathcal{B}'}N_{\mathcal{A}'}$.

Lemma 3.2. *Let $\mathcal{A}$, $\mathcal{B}$ be dg K-categories, and let N be a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module. The following conditions are equivalent:*

- *(Sw1)* ${}_{\widetilde{\mathcal{B}}}N_{\widetilde{\mathcal{A}}} \in \mathrm{per}(\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}})$ *whenever $\widetilde{\mathcal{A}} \to \mathcal{A}$ and $\widetilde{\mathcal{B}} \to \mathcal{B}$ are cofibrant resolutions.*
- *(Sw2)* ${}_{\widetilde{\mathcal{B}}}N_{\widetilde{\mathcal{A}}} \in \mathrm{per}(\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}})$ *for a cofibrant resolution $\widetilde{\mathcal{A}} \to \mathcal{A}$ and a cofibrant resolution $\widetilde{\mathcal{B}} \to \mathcal{B}$.*
- *(Sw3)* ${}_{\mathcal{B}}N_{\widetilde{\mathcal{A}}} \in \mathrm{per}(\widetilde{\mathcal{A}} \otimes_K \mathcal{B}^{\mathrm{op}})$ *for a cofibrant resolution $\widetilde{\mathcal{A}} \to \mathcal{A}$.*
- *(Sw4)* ${}_{\widetilde{\mathcal{B}}}N_{\mathcal{A}} \in \mathrm{per}(\mathcal{A} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}})$ *for a cofibrant resolution $\widetilde{\mathcal{B}} \to \mathcal{B}$.*
- *(Sw5)* ${}_{\mathcal{B}'}N_{\mathcal{A}'} \in \mathrm{per}(\mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}})$ *whenever $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{B}' \to \mathcal{B}$ are K-h-flat resolutions.*
- *(Sw6)* ${}_{\mathcal{B}'}N_{\mathcal{A}'} \in \mathrm{per}(\mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}})$ *for a K-h-flat resolution $\mathcal{A}' \to \mathcal{A}$ and a K-h-flat resolution $\mathcal{B}' \to \mathcal{B}$.*
- *(Sw7)* ${}_{\mathcal{B}}N_{\mathcal{A}'} \in \mathrm{per}(\mathcal{A}' \otimes_K \mathcal{B}^{\mathrm{op}})$ *for a K-h-flat resolution $\mathcal{A}' \to \mathcal{A}$.*

(Sw8) ${}_{\mathcal{B}'}N_{\mathcal{A}} \in \operatorname{per}(\mathcal{A} \otimes_K \mathcal{B}'^{\mathrm{op}})$ *for a* K*-h-flat resolution* $\mathcal{B}' \to \mathcal{B}$.

Proof. Obviously (Sw5) $\Rightarrow$ (Sw1) $\Rightarrow$ (Sw2) $\Rightarrow$ (Sw6), and (Sw3) $\Rightarrow$ (Sw7), and (Sw4) $\Rightarrow$ (Sw8).

Let $\widetilde{a} : \widetilde{\mathcal{A}} \to \mathcal{A}$ be a cofibrant resolution and $a' : \mathcal{A}' \to \mathcal{A}$ a K-h-flat resolution. Then there is a quasi-equivalence $h : \widetilde{\mathcal{A}} \to \mathcal{A}'$ such that $a'h = \widetilde{a}$ (Lemma 3.1). Let $\widetilde{b} : \widetilde{\mathcal{B}} \to \mathcal{B}$ be a cofibrant resolution. In the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{A}' \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}} \arrow[d] \\
\widetilde{\mathcal{A}} \otimes_K \mathcal{B}^{\mathrm{op}} \arrow[r] & \mathcal{A}' \otimes_K \mathcal{B}^{\mathrm{op}}
\end{tikzcd}$$

all morphisms are quasi-equivalences, by Lemma 2.15 and the 3-out-of-2-property. This shows (using equivalence (2.13)) that (Sw1) $\Leftrightarrow$ (Sw3) $\Leftrightarrow$ (Sw7). The proof of (Sw1) $\Leftrightarrow$ (Sw4) $\Leftrightarrow$ (Sw8) is similar.

Let $b' : \mathcal{B}' \to \mathcal{B}$ be a K-h-flat resolution. There is a quasi-equivalence $l : \widetilde{\mathcal{B}} \to \mathcal{B}'$ such that $b'l = \widetilde{b}$ (Lemma 3.1). In the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{A}' \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}} \arrow[d] \\
\widetilde{\mathcal{A}} \otimes_K \mathcal{B}'^{\mathrm{op}} \arrow[r] & \mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}}
\end{tikzcd}$$

all morphisms are quasi-equivalences, by Lemma 2.15. This proves (Sw6) $\Leftrightarrow$ (Sw1) as well as (Sw1) $\Leftrightarrow$ (Sw5). $\square$

Definition 3.3. *A dg* $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$*-module* N *is **good** (more precisely* K***-good**) if it satisfies the equivalent conditions of Lemma 3.2. (Note that one can rewrite all these conditions using equality* (2.4)*.)*

Lemma 3.4. *If* $\mathcal{R} \to \mathcal{A}$ *and* $\mathcal{S} \to \mathcal{B}$ *are quasi-equivalences and* N *is a dg* $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$*-module, then* N *is* K*-good if and only if* ${}_{\mathcal{S}}N_{\mathcal{R}}$ *is* K*-good.*

Proof. Let $\mathcal{A}' \to \mathcal{A}$ and $\mathcal{B}' \to \mathcal{B}$ be K-h-flat resolutions, and let $\widetilde{\mathcal{R}} \to \mathcal{R}$ and $\widetilde{\mathcal{S}} \to \mathcal{S}$ be cofibrant resolutions. Lemma 3.1 shows that the quasi-equivalences $\widetilde{\mathcal{R}} \to \mathcal{R} \to \mathcal{A}$ and $\widetilde{\mathcal{S}} \to \mathcal{S} \to \mathcal{B}$ lift to quasi-equivalences $\widetilde{\mathcal{R}} \to \mathcal{A}'$ and $\widetilde{\mathcal{S}} \to \mathcal{B}'$ respectively. These lifts give rise to the commutative diagram

$$\begin{tikzcd}
\widetilde{\mathcal{R}} \otimes_K \widetilde{\mathcal{S}}^{\mathrm{op}} \arrow[r] \arrow[d] & \mathcal{A}' \otimes_K \widetilde{\mathcal{S}}^{\mathrm{op}} \arrow[d] \\
\widetilde{\mathcal{R}} \otimes_K \mathcal{B}'^{\mathrm{op}} \arrow[r] & \mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}}
\end{tikzcd}$$

of quasi-equivalences (Lemma 2.15). Hence ${}_{\mathcal{B}'}N_{\mathcal{A}'} \in \operatorname{per}(\mathcal{A}' \otimes_K \mathcal{B}'^{\mathrm{op}})$ if and only if ${}_{\widetilde{\mathcal{S}}}N_{\widetilde{\mathcal{R}}} \in \operatorname{per}(\widetilde{\mathcal{R}} \otimes_K \widetilde{\mathcal{S}}^{\mathrm{op}})$. This proves the claim. □

By a dg K-algebra we mean a dg K-category with a unique object, and we sometimes just refer to the endomorphism space of this object.

Corollary 3.5. *Let $\mathcal{A}$ be a dg K-category and B a dg K-algebra. Let N be a dg $\mathcal{A} \otimes_K B^{\mathrm{op}}$-module. Assume that the structure morphism $K \to B$ is a quasi-isomorphism. Then N is K-good if and only if $N_{\mathcal{A}} := \operatorname{res}_{\mathcal{A}}^{\mathcal{A}\otimes_K B^{\mathrm{op}}}(N) \in \operatorname{per}(\mathcal{A})$ (restriction along $\mathcal{A} \xrightarrow{\sim} \mathcal{A} \otimes_K K \to \mathcal{A} \otimes_K B^{\mathrm{op}}$).*

Proof. Apply the lemma to $\mathcal{R} = \mathcal{A}$ and $\mathcal{S} = K \to \mathcal{B} = B$. □

Any ring R can be viewed as an R-R-bimodule ("diagonal bimodule"). Similarly, any dg K-category $\mathcal{A}$ gives rise to the dg $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}$-module ($\mathcal{A}$-$\mathcal{A}$-bimodule) $\mathcal{A}$ whose action morphisms

$$\mathcal{A}(A'', A''') \otimes_K \mathcal{A}(A', A'') \otimes_K \mathcal{A}(A, A') \to \mathcal{A}(A, A'''),\qquad f \otimes g \otimes h \to f.g.h := f \circ g \circ h, \tag{3.1}$$

are just given by composition (where on the left we formally have to move $\mathcal{A}(A'', A''')$ as $\mathcal{A}^{\mathrm{op}}(A''', A'')$ to the right). We call this module the **diagonal bimodule**.

Lemma 3.6. *Let $\mathcal{A}$ be a dg K-category. The following conditions are equivalent:*

(Sm1) $\widetilde{\mathcal{A}} \in \operatorname{per}(\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}})$ for every cofibrant resolution $\widetilde{\mathcal{A}} \to \mathcal{A}$.
(Sm2) $\widetilde{\mathcal{A}} \in \operatorname{per}(\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}})$ for a cofibrant resolution $\widetilde{\mathcal{A}} \to \mathcal{A}$.
(Sm3) $\mathcal{A}' \in \operatorname{per}(\mathcal{A}' \otimes_K \mathcal{A}'^{\mathrm{op}})$ for every K-h-flat resolution $\mathcal{A}' \to \mathcal{A}$.
(Sm4) $\mathcal{A}' \in \operatorname{per}(\mathcal{A}' \otimes_K \mathcal{A}'^{\mathrm{op}})$ for a K-h-flat resolution $\mathcal{A}' \to \mathcal{A}$.
(Sm5) The diagonal bimodule $\mathcal{A}$ is K-good.

Proof. This is a consequence of Lemma 3.2 applied to the diagonal bimodule $\mathcal{A}$, and the following observation: Any morphism of dg K-categories $F : \mathcal{R} \to \mathcal{A}$ gives rise to a morphism from the diagonal bimodule $\mathcal{R}$ to the restriction ${}_{\mathcal{R}}\mathcal{A}_{\mathcal{R}}$ of the diagonal bimodule $\mathcal{A}$, which is a quasi-isomorphism if and only if F induces quasi-isomorphisms on morphism spaces. □

Definition 3.7 (cf. e. g. [Toë09, Def. 2.3])**.** *A (small) dg K-category $\mathcal{A}$ is **smooth** (more precisely K**-smooth**) if it satisfies the equivalent conditions of Lemma 3.6. (Note that one can rewrite these conditions using equality* (2.4)*.)*

Remark 3.8. *In practice conditions (Sm3) and (Sm4) are useful for (dis)proving smoothness. For example a K-h-flat dg K-category $\mathcal{A}$ is smooth if and only if $\mathcal{A} \in \operatorname{per}(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}})$.*

More concretely, if a dg K-algebra A is free (or semi-free or cofibrant or K-h-flat) when considered as a dg K-module, then A is smooth if and only if $A \in \operatorname{per}(A \otimes_K A)$.

Remark 3.9. *If k is a field considered as a dg algebra concentrated in degree zero, then any dg k-module is cofibrant (since it is isomorphic to a coproduct of shifts of k and shifts of* $\operatorname{Cone}(\mathrm{id}_{\mathsf{k}})$ *) and hence (Lemma 2.8)) k-h-flat. In particular any dg k-category is k-h-flat. Hence a dg k-category $\mathcal{A}$ is smooth if and only if $\mathcal{A} \in \operatorname{per}(\mathcal{A} \otimes_{\mathsf{k}} \mathcal{A}^{\mathrm{op}})$.*

This shows that our definition of smoothness generalizes the usual notion of smoothness over a field (see e. g. [Lun10, Def. 3.1]*).*

Examples 3.10. *Consider $\mathbb{C}[X]$ as a dg $\mathbb{C}$-algebra with X of positive degree and differential zero. Then we have:*

(a) $\mathbb{C}[X]$ is $\mathbb{C}$-smooth.
(b) $\mathbb{C}[X]/(X^n)$ is not $\mathbb{C}$-smooth for $n \geq 2$, cf. Proposition 3.40 below.

Assume now that X has positive even degree, so that $\mathbb{C}[X]$ is graded commutative.

(c) $\mathbb{C}[X]/(X^n)$ is not $\mathbb{C}[X]$-smooth, for $n \geq 1$, cf. Proposition 3.43 below. Note that $\mathbb{C} \in \operatorname{per}(\mathbb{C} \otimes_{\mathbb{C}[X]} \mathbb{C}) = \operatorname{per}(\mathbb{C})$, so $\mathbb{C}[X]$-smoothness of $\mathbb{C}$ cannot be checked naively (without a suitable resolution).

Remark 3.11. *We claim that the opposite of a smooth dg K-category is smooth. This follows from the following observations. If $\widetilde{\mathcal{R}} \to \mathcal{R}$ is a cofibrant resolution, then $\widetilde{\mathcal{R}}^{\mathrm{op}} \to \mathcal{R}^{\mathrm{op}}$ is a cofibrant resolution. If $\mathcal{A}$ and $\mathcal{B}$ are dg K-categories, there is an obvious isomorphism $\mathcal{A} \otimes_K \mathcal{B} \xrightarrow{\sim} \mathcal{B} \otimes_K \mathcal{A}$ of dg K-categories. By restriction it induces an isomorphism $D(\mathcal{B} \otimes_K \mathcal{A}) \xrightarrow{\sim} D(\mathcal{A} \otimes_K \mathcal{B})$ which of course preserves compact objects. For $\mathcal{B} = \mathcal{A}^{\mathrm{op}}$ it sends the diagonal bimodule $\mathcal{A}$ to the diagonal bimodule $\mathcal{A}^{\mathrm{op}}$. These statements prove the claim.*

Recall that two dg K-categories are quasi-equivalent if they can be connected by a zig-zag of quasi-equivalences.

Lemma 3.12 (Invariance of smoothness under quasi-equivalence)**.** *If $\mathcal{A} \to \mathcal{B}$ is a quasi-equivalence, then $\mathcal{A}$ is smooth if and only if $\mathcal{B}$ is smooth.*

In particular, if two dg K-categories are quasi-equivalent, then they are either both smooth or both not smooth.

Proof. This follows from Lemma 3.4 (and the observation in the proof of Lemma 3.6). □

3.1. **Invariance of smoothness under dg Morita equivalence.**

Definition 3.13 (cf. [Kel06, 3.8])**.** *Two dg K-categories $\mathcal{A}$, $\mathcal{B}$ are* ***dg Morita equivalent*** *if $D(\mathcal{A})$ and $D(\mathcal{B})$ are connected by a zig-zag of tensor equivalences (as defined after* (2.11)*).*

The aim of this section is to prove Theorem 3.17 below which says that smoothness is invariant under dg Moria equivalence.

Lemma 3.14. *Let $\mathcal{A}$ be a dg K-category and let $b : \widetilde{\mathcal{B}} \to \mathcal{B}$ be a trivial fibration in* dgcat_K*. Let $X = {}_{\mathcal{B}}X_{\mathcal{A}}$ be a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module, and let X' be its restriction to a dg $\mathcal{A} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}}$-module. Then the diagram*

$$\begin{tikzcd} D(\widetilde{\mathcal{B}}) \arrow[r, "?\otimes^L_{\widetilde{\mathcal{B}}} X'"] & D(\mathcal{A}) \\ D(\mathcal{B}) \arrow[u, "\mathrm{res}^{\mathcal{B}}_{\widetilde{\mathcal{B}}}"] \arrow[ur, "?\otimes^L_{\mathcal{B}} X"'] & \end{tikzcd}$$

commutes up to a natural isomorphism.

Proof. **Step 1:** (In this step it is sufficient to assume that b is epimorphic on objects and morphisms). We claim that the obvious evaluation morphism

$$\mathcal{B} \otimes_{\widetilde{\mathcal{B}}} X' \to X \tag{3.2}$$

is an isomorphism of dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-modules. (This generalizes $R/I \otimes_R M = M$ for M an R/I-module.)

The evaluation of $\mathcal{B} \otimes_{\widetilde{\mathcal{B}}} X'$ at $(A, B) \in \mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$ is (by the definition of the tensor product) the cokernel of the obvious morphism

$$\beta : \bigoplus_{\widetilde{B}', \widetilde{B}'' \in \widetilde{\mathcal{B}}} \mathcal{B}(b\widetilde{B}'', B) \otimes_K \widetilde{\mathcal{B}}(\widetilde{B}', \widetilde{B}'') \otimes_K X(A, b\widetilde{B}') \to \bigoplus_{\widetilde{B} \in \widetilde{\mathcal{B}}} \mathcal{B}(b\widetilde{B}, B) \otimes_K X(A, b\widetilde{B}).$$

The evaluation map from the object on the right to $X(A, B)$ factors through the cokernel to a morphism

$$e : (\mathcal{B} \otimes_{\widetilde{\mathcal{B}}} X')(A, B) \to X(A, B).$$

We need to show that e is an isomorphism.

Since b is surjective on objects (trFib2), there is an object $\overline{B} \in \widetilde{B}$ such that $b\overline{B} = B$. Let s be the composition

$$X(A, B) \to \mathcal{B}(b\overline{B}, B) \otimes_K X(A, b\overline{B}) \hookrightarrow \bigoplus_{\widetilde{B} \in \widetilde{\mathcal{B}}} \mathcal{B}(b\widetilde{B}, B) \otimes_K X(A, b\widetilde{B}) \twoheadrightarrow (\mathcal{B} \otimes_{\widetilde{\mathcal{B}}} X')(A, B)$$

where the first map is defined by $x \mapsto \mathrm{id}_B \otimes x$, the second map is the canonical inclusion and the third map is the projection onto the cokernel. Then obviously $es = \mathrm{id}$. Hence e is surjective and it is enough to show that s is surjective. Let $\widetilde{B} \in \widetilde{\mathcal{B}}$ and $f \otimes x \in \mathcal{B}(b\widetilde{B}, B) \otimes_K X(A, b\widetilde{B})$ be a pure tensor. Since b is surjective on morphism spaces there is an element $\overline{f} \in \widetilde{\mathcal{B}}(\widetilde{B}, \overline{B})$ such that $b(\overline{f}) = f$. Then β maps the element

$$\mathrm{id}_B \otimes \overline{f} \otimes x \in \mathcal{B}(b\overline{B}, B) \otimes_K \widetilde{\mathcal{B}}(\widetilde{B}, \overline{B}) \otimes_K X(A, b\widetilde{B})$$

to $f \otimes x - \mathrm{id}_B \otimes fx$. This implies that s is surjective and proves our claim that (3.2) is an isomorphism.

Step 2: If Y is a dg $\widetilde{\mathcal{B}} \otimes_K \mathcal{B}^{\mathrm{op}}$-module, there is an obvious natural transformation

$$(? \otimes^L_{\widetilde{\mathcal{B}}} X') \circ (? \otimes^L_{\mathcal{B}} Y) \to (? \otimes^L_{\mathcal{B}} (Y \otimes_{\widetilde{\mathcal{B}}} X'))$$

of functors $D(\mathcal{B}) \to D(\mathcal{A})$. Putting $Y = \mathcal{B} = {}_{\mathcal{B}}\mathcal{B}_{\widetilde{\mathcal{B}}}$ and using the isomorphism (3.2) we obtain a natural transformation

$$\tau : (? \otimes^L_{\widetilde{\mathcal{B}}} X') \circ (? \otimes^L_{\mathcal{B}} \mathcal{B}) \to (? \otimes^L_{\mathcal{B}} X)$$

Since obviously $(? \otimes^L_{\mathcal{B}} \mathcal{B}) \xrightarrow{\sim} \mathrm{res}^{\mathcal{B}}_{\widetilde{\mathcal{B}}}$ it is enough to show that τ is an isomorphism.

Note that τ is a natural transformation of triangulated functors that commute with coproducts, and recall that $D(\mathcal{B})$ is the localizing subcategory of $D(\mathcal{B})$ generated by the objects $\widehat{B}$, for $B \in \mathcal{B}$, (cf. after (2.4)). Hence to show that τ is an isomorphism it is sufficient to show that $\tau_{\widehat{B}}$ is an isomorphism for all $B \in \mathcal{B}$.

Let $B \in \mathcal{B}$. Since $\widehat{B}$ is cofibrant we have $\widehat{B} \otimes^L_{\mathcal{B}} X \cong \widehat{B} \otimes_{\mathcal{B}} X = X(?, B)$ in $D(\mathcal{A})$ and $\widehat{B} \otimes^L_{\mathcal{B}} \mathcal{B} \cong \widehat{B} \otimes_{\mathcal{B}} \mathcal{B} = \mathcal{B}(b?, B)$ in $D(\widetilde{B})$. Since b is surjective on objects (trFib2), there is an object $\widetilde{B}$ such that $b\widetilde{B} = B$. Then $\widehat{\widetilde{B}} = \widetilde{\mathcal{B}}(?, \widetilde{B}) \xrightarrow{b} \mathcal{B}(b?, B)$ is a cofibrant resolution by (trFib1) and Theorem 2.2. Using this we have

$$(\widehat{B} \otimes^L_{\mathcal{B}} \mathcal{B}) \otimes^L_{\widetilde{\mathcal{B}}} X' \cong \widehat{\widetilde{B}} \otimes_{\widetilde{\mathcal{B}}} X' = X'(?, \widetilde{B}) = X(?, b\widetilde{B}) = X(?, B)$$

in $D(\mathcal{A})$, and under this identifications $\tau_{\widehat{B}}$ is the identity of $X(?, B)$. □

Corollary 3.15. *Let $\mathcal{A}$ and $\mathcal{B}$ be dg K-categories. Assume that $X = {}_{\mathcal{B}}X_{\mathcal{A}}$ is a dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module such that the functor $LT_X := (? \otimes^L_{\mathcal{B}} X) : D(\mathcal{B}) \to D(\mathcal{A})$ is an equivalence.*

Let $a : \widetilde{\mathcal{A}} \to \mathcal{A}$ and $b : \widetilde{\mathcal{B}} \to \mathcal{B}$ be cofibrant resolutions, and let $\widetilde{X}$ be the $\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}}$-module obtained by restriction from X. Then $LT_{\widetilde{X}} := (? \otimes^L_{\widetilde{\mathcal{B}}} \widetilde{X}) : D(\widetilde{\mathcal{B}}) \to D(\widetilde{\mathcal{A}})$ is an equivalence.

Proof. Let $X_{\widetilde{\mathcal{A}}}$ be X viewed as an $\widetilde{\mathcal{A}}\otimes_K \mathcal{B}^{\mathrm{op}}$-module and consider the following diagram.

$$\begin{tikzcd}
D(\widetilde{\mathcal{B}}) \arrow[r, "?\otimes^L_{\widetilde{\mathcal{B}}}\widetilde{X}"] & D(\widetilde{\mathcal{A}}) \\
D(\mathcal{B}) \arrow[u, "\mathrm{res}^{\mathcal{B}}_{\widetilde{\mathcal{B}}}"] \arrow[r, "?\otimes^L_{\mathcal{B}}X"'] \arrow[ur, "?\otimes^L_{\mathcal{B}}X_{\widetilde{\mathcal{A}}}" description] & D(\mathcal{A}) \arrow[u, "\mathrm{res}^{\mathcal{A}}_{\widetilde{\mathcal{A}}}"']
\end{tikzcd}$$

Its lower right triangle is obviously commutative. Its upper left triangle is commutative up to a natural isomorphism by Lemma 3.14. The assumptions imply (cf. (2.12)) that both vertical functors and the lower horizontal functor are equivalences. Hence the remaining two arrows are equivalences. □

Proposition 3.16. *Let $\mathcal{A}$ and $\mathcal{B}$ be cofibrant dg K-categories. Let X' be a dg $\mathcal{A}\otimes_K\mathcal{B}^{\mathrm{op}}$-module such that the functor $LT_{X'} := (? \otimes^L_{\mathcal{B}} X') : D(\mathcal{B}) \to D(\mathcal{A})$ is an equivalence. Then $\mathcal{A}$ is smooth if and only if $\mathcal{B}$ is smooth.*

Proof. The main argument of this proof is from [Lun10, Lemma 3.9]. Some technical details are extracted from [Kel94, Section 6].

By Lemma 3.6 we have to show that $\mathcal{A} \in \mathrm{per}(\mathcal{A}\otimes_K\mathcal{A}^{\mathrm{op}})$ if and only if $\mathcal{B} \in \mathrm{per}(\mathcal{B}\otimes_K\mathcal{B}^{\mathrm{op}})$.

Let $\xi : X \to X'$ be a cofibrant resolution in $C(\mathcal{A}\otimes_K \mathcal{B}^{\mathrm{op}})$. Let N be a dg $\mathcal{B}$-module and consider the diagram

$$\begin{tikzcd}
LT_{X'}(N) \arrow[r, equal] \arrow[drr, "\varphi"'] & T_{X'}pN \arrow[d] & T_XpN \arrow[l, "T_\xi", "\sim"'] \arrow[r, equal] \arrow[d, "\sim"] & LT_XN \\
 & T_{X'}N & T_XN \arrow[l, "T_\xi"] &
\end{tikzcd} \tag{3.3}$$

in $D(\mathcal{A})$ with obvious vertical and horizontal morphisms. The upper horizontal arrow is an isomorphism since the cofibrant dg $\mathcal{B}$-module pN is $\mathcal{B}$-h-flat (Lemma 2.8). The right vertical arrow is an isomorphism since X is $\mathcal{B}^{\mathrm{op}}$-h-flat (obvious variant of Prop. 2.10, part (b)). We define $\varphi = \varphi_N$ to be the indicated composition of these isomorphisms. In fact this extends to a natural isomorphism $\varphi : LT_{X'} \xrightarrow{\sim} T_X$ of functors $D(\mathcal{B}) \to D(\mathcal{A})$ (where T_X is defined in the obvious way, using that X is $\mathcal{B}^{\mathrm{op}}$-h-flat). In particular T_X is an equivalence.

For any $B \in \mathcal{B}$, the dg $\mathcal{A}$-module $X^B = X(?, B)$ is cofibrant by Proposition 2.10 and in particular h-projective (Lemma 2.6). This implies that H_X as defined in (2.9) maps acyclic dg $\mathcal{A}$-modules to acyclic dg $\mathcal{B}$-modules, and hence descends directly to a triangulated functor $H_X : D(\mathcal{A}) \to D(\mathcal{B})$. The unit ε and counit η of the adjunction (2.8) hence directly provide an adjunction (T_X, H_X) between $T_X : D(\mathcal{B}) \to D(\mathcal{A})$ and

$H_X : D(\mathcal{A}) \to D(\mathcal{B})$. Since T_X is an equivalence, H_X is a quasi-inverse. Since T_X preserves all coproducts, the same is true for H_X.

Recall the definition of the dg $\mathcal{B} \otimes_K \mathcal{A}$-module $X^\perp$ from (2.10) and that there is a canonical transformation $\tau : T_{X^\perp} \to H_X$. This morphism provides a natural transformation $\widetilde{\tau} : LT_{X^\perp} \to H_X$ of triangulated functors, defined on an object $M \in D(\mathcal{A})$ as the indicated composition in the following commutative diagram.

$$\begin{tikzcd}
LT_{X^\perp}M \arrow[r, equal] \arrow[drr, "\widetilde{\tau}"'] & T_{X^\perp}pM \arrow[r, "\tau"] \arrow[d] & H_X(pM) \arrow[d, "\sim"] \\
& T_{X^\perp}M \arrow[r, "\tau"] & H_X(M)
\end{tikzcd} \tag{3.4}$$

It is clear that the vertical morphism on the right is an isomorphism, and it is easy to check that the upper horizontal arrow is an isomorphism if $M = \widehat{A}$, for all $A \in \mathcal{A}$; hence all $\widetilde{\tau}_{\widehat{A}}$ are isomorphisms. Since both $LT_{X^\perp}$ and H_X preserve all coproducts, this implies already (by the same argument as in Step 2 of the proof of Lemma 3.14) that $\widetilde{\tau}$ is a natural isomorphism (and that the upper horizontal arrow in diagram (3.4) is an isomorphism for all M).

Let $\upsilon : Y \to X^\perp$ be a cofibrant resolution of $X^\perp$ in $C(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$. As above (cf. (3.3)) we explicitly construct an isomorphism $\psi : LT_{X^\perp} \xrightarrow{\sim} T_Y$.

Note that $\psi \circ \widetilde{\tau}^{-1} : H_X \xrightarrow{\sim} T_Y$ is an isomorphism which shows that T_X has a quasi-inverse given by a tensor-functor.

For N a dg $\mathcal{B}$-module consider the following commutative diagram in $D(\mathcal{B})$ which is built from the adjunction morphism ε_N, from (3.4) and the analog of (3.3).

$$\begin{tikzcd}
N \arrow[r, "\varepsilon_N", "\sim"'] & H_X T_X N & T_{X^\perp} T_X N \arrow[l, "\tau"'] & T_Y T_X N \arrow[l, "T_\upsilon"'] \arrow[ll, dotted, bend right, "\tau \circ T_\upsilon"', "\sim"] \\
& H_X p T_X N \arrow[u, "\sim"] & T_{X^\perp} p T_X N \arrow[u] \arrow[l, "\tau"', "\sim"] & T_Y p T_X N \arrow[u, "\sim"'] \arrow[l, "T_\upsilon"', "\sim"] \\
& & LT_{X^\perp} T_X N \arrow[u, equal] \arrow[uul, "\widetilde{\tau}"] \arrow[uur, "\psi"'] &
\end{tikzcd}$$

It implies that the dotted composition $\tau \circ T_\upsilon : T_Y T_X N \to H_X T_X N$ in the upper row is (as indicated) an isomorphism. This is important for the following reason: If N has additionally a left dg $\mathcal{R}$-module structure (i. e. it is an dg $\mathcal{B} \otimes_K \mathcal{R}^{\mathrm{op}}$-module) then all morphisms in the upper row are morphisms of dg $\mathcal{B} \otimes_K \mathcal{R}^{\mathrm{op}}$-modules and in fact isomorphisms in

$D(\mathcal{B} \otimes_K \mathcal{R}^{\mathrm{op}})$ since we can test this by plugging in $R \in \mathcal{R}$. (A priori the entries in the lower row have no left dg $\mathcal{R}$-module structure.)

We apply this to $\mathcal{R} = \mathcal{B}$ and the diagonal bimodule $\mathcal{B}$ and obtain isomorphisms in $D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$ (or quasi-isomorphisms in $C(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$ or $\mathcal{H}(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$)

$$\mathcal{B} \xrightarrow[\sim]{\varepsilon_{\mathcal{B}}} H_X T_X \mathcal{B} \xleftarrow[\sim]{\tau \circ T_\upsilon} T_Y T_X \mathcal{B} = \mathcal{B} \otimes_{\mathcal{B}} X \otimes_{\mathcal{A}} Y = X \otimes_{\mathcal{A}} Y \tag{3.5}$$

Similar we obtain for M a dg $\mathcal{A}$-module the following diagram in $D(\mathcal{A})$.

$$\begin{tikzcd}
M & T_X H_X M \arrow[l, "\eta_M", "\sim"'] & T_X T_{X^\perp} M \arrow[l, "T_X\tau"'] & T_X T_Y M \arrow[l, "T_X T_\upsilon"'] \arrow[ll, bend right, dotted, "T_X\tau \circ T_X T_\upsilon"', "\sim"] \\
 & T_X H_X pM \arrow[u, "\sim"] & T_X T_{X^\perp} pM \arrow[l, "T_X\tau", "\sim"'] \arrow[u] & T_X T_Y pM \arrow[l, "T_X T_\upsilon", "\sim"'] \arrow[u, "\sim"'] \\
 & & T_X L T_{X^\perp} M \arrow[u, equal] \arrow[uul, "T_X\widetilde{\tau}"] \arrow[uur, "T_X\psi"'] &
\end{tikzcd}$$

The upper row is again compatible with any available left dg module structure on M. Applied to the diagonal bimodule $\mathcal{A}$ we obtain an isomorphism

$$\mathcal{A} \xleftarrow[\sim]{\eta_{\mathcal{A}}} T_X H_X \mathcal{A} \xleftarrow[\sim]{T_X\tau \circ T_X T_\upsilon} T_X T_Y \mathcal{A} = \mathcal{A} \otimes_{\mathcal{A}} Y \otimes_{\mathcal{B}} X = Y \otimes_{\mathcal{B}} X \tag{3.6}$$

in $D(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}})$.

The dg K-functor

$$_Y\Delta_X(?) := Y \otimes_{\mathcal{B}} ? \otimes_{\mathcal{B}} X : \mathcal{M}od(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}) \to \mathcal{M}od(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}})$$

is the composition of the two dg K-functors $(Y \otimes_{\mathcal{B}} ?)$ and $(? \otimes_{\mathcal{B}} X)$; it preserves acyclic modules since Y is $\mathcal{B}$-flat and X is $\mathcal{B}^{\mathrm{op}}$-flat (Prop. 2.10, part (b)). Hence it directly descends to a triangulated functor

$$_Y\Delta_X = Y \otimes_{\mathcal{B}} ? \otimes_{\mathcal{B}} X : D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}) \to D(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}).$$

Similarly, we define a functor

$$_X\Delta_Y = X \otimes_{\mathcal{A}} ? \otimes_{\mathcal{A}} Y : D(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}) \to D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}).$$

We claim that $_Y\Delta_X$ and $_X\Delta_Y$ are quasi-inverse to each other. This follows from (3.5) and (3.6) but let us include the details: The functor $_X\Delta_Y \circ {}_Y\Delta_X$ coincides with the obvious composition

$$D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}) \xrightarrow{X \otimes_{\mathcal{A}} Y \otimes_{\mathcal{B}} ?} D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}) \xrightarrow{? \otimes_{\mathcal{B}} X \otimes_{\mathcal{A}} Y} D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}).$$

The second functor is (canonically isomorphic to) the functor $LT_{X\otimes_{\mathcal{A}}Y}$. The morphisms in (3.5) are quasi-isomorphism when considered in $C(\mathcal{B}\otimes_K \mathcal{B}^{\mathrm{op}})$. Hence they induce an isomorphism between $LT_{X\otimes_{\mathcal{A}}Y}$ and $LT_{\mathcal{B}} = \mathrm{id}$ (use Lemma 2.8). A similar reasoning applies to the first functor, and hence ${}_X\Delta_Y \circ {}_Y\Delta_X \cong \mathrm{id}$. Similarly, (3.6) implies ${}_Y\Delta_X \circ {}_X\Delta_Y \cong \mathrm{id}$.

The mutually quasi-inverse equivalences ${}_Y\Delta_X$ and ${}_X\Delta_Y$ preserve compact objects. Using (3.6) again we have ${}_Y\Delta_X(\mathcal{B}) = Y\otimes_{\mathcal{B}}\mathcal{B}\otimes_{\mathcal{B}} X = Y\otimes_{\mathcal{B}} X \cong \mathcal{A}$, and similarly ${}_X\Delta_Y(\mathcal{A})\cong\mathcal{B}$. This implies that the diagonal bimodule $\mathcal{A}$ is compact if and only if the diagonal bimodule $\mathcal{B}$ is compact. □

Theorem 3.17. *Let $\mathcal{A}$ and $\mathcal{B}$ be dg K-categories. If $\mathcal{A}$ and $\mathcal{B}$ are dg Morita equivalent, then $\mathcal{A}$ is smooth if and only if $\mathcal{B}$ is smooth.*

Proof. It is enough to show the claim if there is a tensor equivalence $D(\mathcal{B}) \xrightarrow{\sim} D(\mathcal{A})$. Let $\widetilde{\mathcal{A}} \to \mathcal{A}$ and $\widetilde{B} \to \mathcal{B}$ be cofibrant resolutions. By Corollary 3.15 we can lift our tensor equivalence to a tensor equivalence $D(\widetilde{\mathcal{B}}) \xrightarrow{\sim} D(\widetilde{\mathcal{A}})$. Now the result follows from Proposition 3.16 and Lemma 3.12 □

Corollary 3.18. *Let $\mathcal{B}\to\mathcal{A}$ be a morphism in* dgcat_K. *If $\mathrm{res}^{\mathcal{A}}_{\mathcal{B}} : D(\mathcal{A}) \to D(\mathcal{B})$ is an equivalence, then $\mathcal{A}$ is smooth if and only if $\mathcal{B}$ is smooth.*

Proof. Let X be $\mathcal{A}$ viewed as a dg $\mathcal{B}\otimes_K\mathcal{A}^{\mathrm{op}}$-module. Then $T_X = (?\otimes_{\mathcal{A}} X) \xrightarrow{\sim} \mathrm{res}^{\mathcal{A}}_{\mathcal{B}}$, and hence $LT_X \xrightarrow{\sim} \mathrm{res}^{\mathcal{A}}_{\mathcal{B}}$ as functors $D(\mathcal{A})\to D(\mathcal{B})$. □

Corollary 3.19. *Let $\mathcal{A}$ be a dg K-category. Assume that $\mathcal{A}$ is triangulated in the sense that it is pretriangulated and that its homotopy category $[\mathcal{A}]$ is Karoubian (= idempotent complete). Assume that there is an object $E\in\mathcal{A}$ such that E is a classical generator of $[\mathcal{A}]$. Let $\mathcal{A}(E) := \mathcal{A}(E,E)$ be the dg K-algebra of endomorphisms of E. Then*

$$\mathcal{A}(E,?) : [\mathcal{A}] \xrightarrow{\sim} \mathrm{per}(\mathcal{A}(E))$$

is a triangulated equivalence, and moreover $\mathcal{A}$ is K-smooth if and only if $\mathcal{A}(E)$ is K-smooth.

Proof. Let $\mathcal{E}\subset\mathcal{A}$ be the full dg K-subcategory whose unique object is E. The upper horizontal functor in the commutative diagram

$$\begin{tikzcd} {[\mathcal{A}]} \arrow[r] \arrow[dr, "\mathcal{A}(E,?)"'] & \mathrm{per}(\mathcal{A}) \arrow[d, "\mathrm{res}^{\mathcal{A}}_{\mathcal{E}}"] \\ & \mathrm{per}(\mathcal{E}) \end{tikzcd}$$

is induced by the Yoneda embedding. It is an equivalence since $\mathcal{A}$ is triangulated. Since E is a classical generator of $[\mathcal{A}] \xrightarrow{\sim} \operatorname{per}(\mathcal{A})$, both non-horizontal functors in the above diagram are equivalences. It follows (cf. explanations around (2.4), or [Lun10, Lemma 2.12]) that $\operatorname{res}^{\mathcal{A}}_{\mathcal{E}} : D(\mathcal{A}) \to D(\mathcal{E})$ is an equivalence. Now use Corollary 3.18. □

3.2. **Directed dg K-categories.** Our aim is to prove Theorem 3.24 below which generalizes and strengthens [Lun10, Prop. 3.11].

Let $\mathcal{A}$ and $\mathcal{B}$ be two small dg K-categories and let $N = {}_{\mathcal{A}}N_{\mathcal{B}}$ be a dg $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$-module. Let $\mathcal{E}$, symbolically denoted

$$\mathcal{E} = \begin{bmatrix} \mathcal{B} & 0 \\ N & \mathcal{A} \end{bmatrix},$$

be the following dg K-category: Its objects are the disjoint union of the objects of $\mathcal{A}$ and $\mathcal{B}$, and its morphisms are given by

$$\mathcal{E}(A, A') = \mathcal{A}(A, A'), \qquad \mathcal{E}(B, A') = N(B, A'),$$
$$\mathcal{E}(A, B') = 0, \qquad \mathcal{E}(B, B') = \mathcal{B}(B, B'),$$

for objects $A, A' \in \operatorname{Obj} \mathcal{A} \subset \operatorname{Obj} \mathcal{E}$ and $B, B' \in \operatorname{Obj} \mathcal{B} \subset \operatorname{Obj} \mathcal{E}$, and units and compositions are obvious (e. g. for $A \in \mathcal{A}$ and $B, B' \in \mathcal{B}$ composition is given by the action morphism

$$\mathcal{E}(B', A) \otimes_K \mathcal{E}(B, B') = N(B', A) \otimes_K \mathcal{B}(B, B') \to N(B, A) = \mathcal{E}(B, A)$$

of the dg $\mathcal{B}$-module N).

Remark 3.20. *Conversely, if $\mathcal{E}$ is a small dg K-category such that we can split the set of objects into two disjoint subsets, giving rise to full subcategories $\mathcal{A}$ and $\mathcal{B}$, such that $\mathcal{E}(\mathcal{A}, \mathcal{B}) = 0$, then $\mathcal{E} = \left[\begin{smallmatrix} \mathcal{B} & 0 \\ N & \mathcal{A} \end{smallmatrix}\right]$ for $N := \mathcal{E}|_{\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}}$ the indicated restriction of the diagonal bimodule $\mathcal{E}$.*

Remark 3.21. *The quiver picture of $\mathcal{E}$ is $\mathcal{B} \xrightarrow{{}_{\mathcal{A}}N_{\mathcal{B}}} \mathcal{A}$.*

If S is a dg $\mathcal{E}$-module we can restrict it along the obvious inclusions $\mathcal{A} \subset \mathcal{E}$ and $\mathcal{B} \subset \mathcal{E}$ and obtain a dg $\mathcal{A}$-module $S|_{\mathcal{A}}$ and a dg $\mathcal{B}$-module $S|_{\mathcal{B}}$. Furthermore the action morphisms $S(A) \otimes_K \mathcal{E}(B, A) \to S(B)$, for $A \in \mathcal{A}$ and $B \in \mathcal{B}$, induce a morphism

$$\varphi_S : S|_{\mathcal{A}} \otimes_{\mathcal{A}} N \to S|_{\mathcal{B}}$$

in $C(\mathcal{B})$. In this manner we see that a dg $\mathcal{E}$-module S is the same as a triple $(S_{\mathcal{A}}, S_{\mathcal{B}}, \varphi : S_{\mathcal{A}} \otimes_{\mathcal{A}} N \to S_{\mathcal{B}})$ where $S_{\mathcal{A}}$ is a dg $\mathcal{A}$-module, $S_{\mathcal{B}}$ is a dg $\mathcal{B}$-module and φ is a morphism in $C(\mathcal{B})$. We describe such a dg $\mathcal{E}$-module symbolically as $S = \boxed{S_{\mathcal{B}} \xleftarrow{\varphi} S_{\mathcal{A}}.}$

Morphisms of $f : S \to S'$ of dg $\mathcal{E}$-modules are in this description pairs $(f_{\mathcal{B}}, f_{\mathcal{A}})$ where $f_{\mathcal{B}} : S_{\mathcal{B}} \to S'_{\mathcal{B}}$ and $f_{\mathcal{A}} : S_{\mathcal{A}} \to S'_{\mathcal{A}}$ are morphisms of dg $\mathcal{B}$- and $\mathcal{A}$-modules respectively such that $f_{\mathcal{B}} \circ \varphi_S = \varphi_{S'} \circ (f_{\mathcal{A}} \otimes_{\mathcal{A}} \mathrm{id}_N)$. We denote such a morphism symbolically as $f = \boxed{f_{\mathcal{B}}\ f_{\mathcal{A}}}$.

Lemma 3.22. *Let $a : \mathcal{A} \subset \mathcal{E}$ be the obvious inclusion. The functor $a_* = \mathrm{res}^{\mathcal{E}}_{\mathcal{A}} : D(\mathcal{E}) \to D(\mathcal{A})$ maps compact objects to compact objects.*

Proof. This functor has a right adjoint $D(\mathcal{A}) \to D(\mathcal{E})$, defined by $U \mapsto \boxed{0 \longleftarrow U}$, which preserves all coproducts. This implies the statement. ☐

The inclusion $b : \mathcal{B} \subset \mathcal{E}$ defines the dg K-functor $\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}} := (? \otimes_{\mathcal{B}} \mathcal{E}) : \mathcal{M}od(\mathcal{B}) \to \mathcal{M}od(\mathcal{E})$, given by

$$\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}}(V) = \boxed{V \longleftarrow 0}. \tag{3.7}$$

It preserves acyclics and descends to a triangulated functor $b^* := \mathrm{prod}^{\mathcal{E}}_{\mathcal{B}} : D(\mathcal{B}) \to D(\mathcal{E})$. This functor has the right adjoint functor $b_* := \mathrm{res}^{\mathcal{E}}_{\mathcal{B}} : D(\mathcal{E}) \to D(\mathcal{B})$, mapping S as above to $S_{\mathcal{B}}$.

Lemma 3.23. *Let $V \in D(\mathcal{B})$. If $b^*(V) = \mathrm{prod}^{\mathcal{E}}_{\mathcal{B}}(V)$ is compact in $D(\mathcal{E})$, then V is compact in $D(\mathcal{B})$.*

Proof. Obviously $b^* : D(\mathcal{B}) \to D(\mathcal{E})$ commutes with all coproducts. The unit $\mathrm{id} \to b_* b^*$ of the adjunction is an isomorphism, so b^* is fully faithful. This implies that statement. ☐

3.3. **Smoothness of directed dg K-categories.** The following is the quiver picture of $\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$, where we additionally have drawn the quiver of $\mathcal{E}$ on the horizontal axis and that of $\mathcal{E}^{\mathrm{op}}$ on the vertical axis (note that the $\mathcal{B} \otimes_K \mathcal{A}$-module N can be viewed as a $\mathcal{A}^{\mathrm{op}} \otimes_K \mathcal{B}$-module N^{op}):

$$\begin{tikzcd}
\mathcal{B}^{\mathrm{op}} & \mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}} \arrow[r, "N \otimes_K \mathcal{B}^{\mathrm{op}}"] & \mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}} \\
\mathcal{A}^{\mathrm{op}} \arrow[u, "N^{\mathrm{op}}"] & \mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}} \arrow[u, "\mathcal{B} \otimes_K N^{\mathrm{op}}"] \arrow[ur, "N \otimes_K N^{\mathrm{op}}"] \arrow[r, "N \otimes_K \mathcal{A}^{\mathrm{op}}"'] & \mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}} \arrow[u, "\mathcal{A} \otimes_K N^{\mathrm{op}}"'] \\
 & \mathcal{B} \arrow[r, "N"] & \mathcal{A}
\end{tikzcd} \tag{3.8}$$

We can describe dg $\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$-modules in a similar way as explained above for dg $\mathcal{E}$-modules: Let M be such a module (we always view it implicitly as a bimodule). Restriction along the four morphisms of dg K-categories $\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$, $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$, $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$, $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ gives rise to the dg modules

$${}_{\mathcal{B}}M_{\mathcal{B}} := {}^{\mathcal{E}}_{\mathcal{B}}\mathrm{res}^{\mathcal{E}}_{\mathcal{B}}(M), \quad {}_{\mathcal{B}}M_{\mathcal{A}} := {}^{\mathcal{E}}_{\mathcal{B}}\mathrm{res}^{\mathcal{E}}_{\mathcal{A}}(M), \quad {}_{\mathcal{A}}M_{\mathcal{B}} := {}^{\mathcal{E}}_{\mathcal{A}}\mathrm{res}^{\mathcal{E}}_{\mathcal{B}}(M), \quad {}_{\mathcal{A}}M_{\mathcal{A}} := {}^{\mathcal{E}}_{\mathcal{A}}\mathrm{res}^{\mathcal{E}}_{\mathcal{A}}(M).$$

Furthermore the action morphisms of N from the right and from the left give rise to closed degree zero morphisms

$$
\begin{aligned}
{}_{\mathcal{B}}\theta_{\mathcal{AB}} &: ({}_{\mathcal{B}}M_{\mathcal{A}}) \otimes_{\mathcal{A}} N \rightarrow {}_{\mathcal{B}}M_{\mathcal{B}}, & {}_{\mathcal{A}}\theta_{\mathcal{AB}} &: ({}_{\mathcal{A}}M_{\mathcal{A}}) \otimes_{\mathcal{A}} N \rightarrow {}_{\mathcal{A}}M_{\mathcal{B}}, \\
{}_{\mathcal{BA}}\theta_{\mathcal{A}} &: N \otimes_{\mathcal{B}} ({}_{\mathcal{B}}M_{\mathcal{A}}) \rightarrow {}_{\mathcal{A}}M_{\mathcal{A}}, & {}_{\mathcal{BA}}\theta_{\mathcal{B}} &: N \otimes_{\mathcal{B}} ({}_{\mathcal{B}}M_{\mathcal{B}}) \rightarrow {}_{\mathcal{A}}M_{\mathcal{B}}
\end{aligned}
$$

of suitable dg modules, for example the last morphism is a morphism in $C(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$. These morphisms fit in the commutative diagram (since the N-left action and the N-right action commute)

$$
\begin{tikzcd}
N \otimes_{\mathcal{B}} ({}_{\mathcal{B}}M_{\mathcal{B}}) \arrow[d, "{}_{\mathcal{BA}}\theta_{\mathcal{B}}"] & N \otimes_{\mathcal{B}} ({}_{\mathcal{B}}M_{\mathcal{A}}) \otimes_{\mathcal{A}} N \arrow[l, "\mathrm{id} \otimes ({}_{\mathcal{B}}\theta_{\mathcal{AB}})"'] \arrow[d, "({}_{\mathcal{BA}}\theta_{\mathcal{A}}) \otimes \mathrm{id}"] \\
{}_{\mathcal{A}}M_{\mathcal{B}} & ({}_{\mathcal{A}}M_{\mathcal{A}}) \otimes_{\mathcal{A}} N. \arrow[l, "{}_{\mathcal{A}}\theta_{\mathcal{AB}}"']
\end{tikzcd} \tag{3.9}
$$

We conclude that a dg $\mathcal{E} \otimes_{\mathcal{E}}^{\mathrm{op}}$-module M is (equivalent to) the datum

$$
{}_{\mathcal{B}}M_{\mathcal{A}}, {}_{\mathcal{A}}M_{\mathcal{A}}, {}_{\mathcal{B}}M_{\mathcal{B}}, {}_{\mathcal{A}}M_{\mathcal{B}}, {}_{\mathcal{B}}\theta_{\mathcal{AB}}, {}_{\mathcal{A}}\theta_{\mathcal{AB}}, {}_{\mathcal{BA}}\theta_{\mathcal{A}}, {}_{\mathcal{BA}}\theta_{\mathcal{B}},
$$

of dg bimodules and closed degree zero morphisms as above such that (3.9) commutes. It is convenient to describe such a bimodule M symbolically by the diagram

$$
\boxed{
\begin{tikzcd}
{}_{\mathcal{B}}M_{\mathcal{B}} \arrow[d, "{}_{\mathcal{BA}}\theta_{\mathcal{B}}"] & {}_{\mathcal{B}}M_{\mathcal{A}} \arrow[l, "{}_{\mathcal{B}}\theta_{\mathcal{AB}}"'] \arrow[d, "{}_{\mathcal{BA}}\theta_{\mathcal{A}}"] \\
{}_{\mathcal{A}}M_{\mathcal{B}} & {}_{\mathcal{A}}M_{\mathcal{A}}. \arrow[l, "{}_{\mathcal{A}}\theta_{\mathcal{AB}}"']
\end{tikzcd}
}
$$

The diagonal bimodule $\mathcal{E}$ (cf. (3.1)) is given by the diagram

$$
\mathcal{E} = \boxed{
\begin{tikzcd}
\mathcal{B} \arrow[d, "\mathrm{id}"] & 0 \arrow[l] \arrow[d] \\
N & \mathcal{A}. \arrow[l, "\mathrm{id}"']
\end{tikzcd}
} \tag{3.10}
$$

where we identify $\mathcal{A} \otimes_{\mathcal{A}} N = N$ and $N \otimes_{\mathcal{B}} \mathcal{B} = N$.

We described in (3.7) an extension of scalars functor. Similarly, we have extension of scalars functors for bimodules, e. g. the morphism $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}} \rightarrow \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ induces the extensions of scalars functor

$$
{}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{A}} : \mathcal{M}od(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}) \rightarrow \mathcal{M}od(\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}).
$$

A computation shows that it maps a dg $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}$-module X to

$$
({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{A}})(X) = \boxed{
\begin{tikzcd}
0 \arrow[d] & 0 \arrow[l] \arrow[d] \\
X \otimes_{\mathcal{A}} N & X. \arrow[l, "\mathrm{id}"']
\end{tikzcd}
} \tag{3.11}
$$

In particular we can apply this to the diagonal $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}}$-module $\mathcal{A}$ and obtain

$$({}^{\mathcal{E}}_{\mathcal{A}}\operatorname{prod}^{\mathcal{E}}_{\mathcal{A}})(\mathcal{A}) = \begin{tikzcd} 0 \arrow[d] & 0 \arrow[l] \arrow[d] \\ N & \mathcal{A}. \arrow[l, "\mathrm{id}"'] \end{tikzcd} \tag{3.12}$$

Similarly, we can induce along $\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ and obtain for the diagonal bimodule $\mathcal{B}$ that

$$({}^{\mathcal{E}}_{\mathcal{B}}\operatorname{prod}^{\mathcal{E}}_{\mathcal{B}})(\mathcal{B}) = \begin{tikzcd} \mathcal{B} \arrow[d, "\mathrm{id}"] & 0 \arrow[l] \arrow[d] \\ N & 0. \arrow[l] \end{tikzcd} \tag{3.13}$$

Similarly, extension of scalars along $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}} \to \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ applied to N gives

$$({}^{\mathcal{E}}_{\mathcal{A}}\operatorname{prod}^{\mathcal{E}}_{\mathcal{B}})(N) = \begin{tikzcd} 0 \arrow[d] & 0 \arrow[l] \arrow[d] \\ N & 0. \arrow[l] \end{tikzcd} \tag{3.14}$$

Now we can prove the following generalization and strengthening of [Lun10, Prop. 3.11].

Theorem 3.24. *Let $\mathcal{A}$ and $\mathcal{B}$ be dg K-categories, let N be a dg $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$-module, and let*

$$\mathcal{E} = \begin{bmatrix} \mathcal{B} & 0 \\ N & \mathcal{A}. \end{bmatrix}$$

The following conditions are equivalent:

- *(E1) $\mathcal{A}$ and $\mathcal{B}$ are K-smooth and N is K-good.*
- *(E2) $\mathcal{E}$ is K-smooth.*

Remark 3.25. *If A and B are dg K-algebras and N is a dg $B \otimes_K A^{\mathrm{op}}$-module, then we can view A and B as dg K-categories and form the dg K-category $\mathcal{E} = \left[\begin{smallmatrix} B & 0 \\ N & A \end{smallmatrix}\right]$ with two objects as above. On the other hand we can consider the obvious dg K-algebra $E := \left[\begin{smallmatrix} B & 0 \\ N & A \end{smallmatrix}\right]$. Then the obvious dg K-equivalence $\mathcal{M}od(E) \xrightarrow{\sim} \mathcal{M}od(\mathcal{E})$ is given by an $\mathcal{E} \otimes_K E^{\mathrm{op}}$-bimodule, and hence E and $\mathcal{E}$ are dg Morita equivalent. In particular E is K-smooth if and only if $\mathcal{E}$ is K-smooth (Theorem 3.17).*

This remark obviously generalizes to dg K-categories/algebras of the form $\left[\begin{smallmatrix} B & M \\ N & A \end{smallmatrix}\right]$, where M is a dg $A \otimes_K B^{\mathrm{op}}$-module, and also two bigger matrix categories/algebras.

Example 3.26. *Assume that k is a field, and let V be a (dg) k-module. Then the dg k-algebra $\left[\begin{smallmatrix} \mathsf{k} & 0 \\ V & \mathsf{k} \end{smallmatrix}\right]$ is k-smooth if and only if V is finite dimensional.*

Example 3.27. *Consider $\mathbb{C}[X]$ as a dg $\mathbb{C}$-algebra with X of positive even degree. Then the dg $\mathbb{C}[X]$-algebra $\left[\begin{smallmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{C}[X] \end{smallmatrix}\right]$ is not $\mathbb{C}[X]$-smooth since $\mathbb{C}$ is not $\mathbb{C}[X]$-smooth, see Example 3.10 (c).*

Proof. **Step 1:** Reduction to the case that $\mathcal{A}$ and $\mathcal{B}$ are cofibrant dg K-categories and that N is a cofibrant dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module.

Let $\widetilde{\mathcal{A}} \to \mathcal{A}$ and $\widetilde{\mathcal{B}} \to \mathcal{B}$ be cofibrant resolutions and let $\widetilde{N} \to {}_{\widetilde{\mathcal{A}}}N_{\widetilde{\mathcal{B}}}$ be a cofibrant resolution (in $C(\widetilde{\mathcal{B}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}})$) of the restriction ${}_{\widetilde{\mathcal{A}}}N_{\widetilde{\mathcal{B}}} := ({}^{\mathcal{A}}_{\widetilde{\mathcal{A}}}\mathrm{res}^{\mathcal{B}}_{\widetilde{\mathcal{B}}})(N)$ of N. Note that $\widetilde{N} \xrightarrow{\sim} {}_{\widetilde{\mathcal{A}}}N_{\widetilde{\mathcal{B}}}$ in $D(\widetilde{\mathcal{B}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}})$. Then (E1) is by definition equivalent to

$$\widetilde{\mathcal{A}} \in \mathrm{per}(\widetilde{\mathcal{A}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}}), \;\; \widetilde{\mathcal{B}} \in \mathrm{per}(\widetilde{\mathcal{B}} \otimes_K \widetilde{\mathcal{B}}^{\mathrm{op}}), \;\text{ and }\; \widetilde{N} \in \mathrm{per}(\widetilde{\mathcal{B}} \otimes_K \widetilde{\mathcal{A}}^{\mathrm{op}}),$$

which is equivalent to $\widetilde{\mathcal{A}}$, $\widetilde{\mathcal{B}}$ being K-smooth and $\widetilde{N}$ being K-good.

Let

$$\mathcal{E}' = \begin{bmatrix} \widetilde{\mathcal{B}} & 0 \\ \widetilde{N} & \widetilde{\mathcal{A}} \end{bmatrix}.$$

The obvious morphism $\mathcal{E}' \to \mathcal{E}$ of dg K-categories is a trivial fibration (check the conditions (trFib1) and (trFib2) using the description of the trivial fibrations in Theorem 2.2), and even a K-h-flat resolution since $\mathcal{E}'$ is K-h-flat (use Lemma 2.14 and Proposition 2.10, part (c)). Hence (E2) is by definition equivalent to $\mathcal{E}' \in \mathrm{per}(\mathcal{E}' \otimes_K \mathcal{E}'^{\mathrm{op}})$, i. e. to K-smoothness of $\mathcal{E}'$.

Step 2: Assume $\mathcal{A}$ and $\mathcal{B}$ are cofibrant dg K-categories and that N is a cofibrant dg $\mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}}$-module. Let $\mathcal{E} = \left[\begin{smallmatrix} \mathcal{B} & 0 \\ N & \mathcal{A} \end{smallmatrix}\right]$. By Step 1 it is enough to prove that (E1) and (E2) are equivalent under these additional assumptions.

Since N is a cofibrant dg $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$-module, the functor ${}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{A}}$ preserves acyclics (use (3.11) and the fact that N is $\mathcal{A}^{\mathrm{op}}$-h-flat by Prop. 2.10, part (b) (and Lemma 2.14)) and hence trivially descends (without taking cofibrant/choosing h-projective resolutions) to a functor between the derived categories denoted by the same symbol. The analog statement holds for ${}^{\mathcal{E}}_{\mathcal{B}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}}$ (similar proof) and for ${}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}}$ (obvious from the fact that it maps Z to $\boxed{\begin{smallmatrix} 0 & 0 \\ Z & 0 \end{smallmatrix}}$, as in (3.14) for $Z = N$). These extension of scalars functors preserve compact objects since restriction, their right adjoints, obviously commute with all coproducts.

The short exact sequence

$$0 \to \boxed{\begin{array}{ccc} 0 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ N & \leftarrow & 0. \end{array}} \xrightarrow{\begin{bmatrix} \boxed{\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}} \\ \boxed{\begin{smallmatrix} 0 & 0 \\ -1 & 0 \end{smallmatrix}} \end{bmatrix}} \boxed{\begin{array}{ccc} 0 & \leftarrow & 0 \\ \downarrow & & \downarrow \\ N & \xleftarrow{\mathrm{id}} & \mathcal{A}. \end{array}} \oplus \boxed{\begin{array}{ccc} \mathcal{B} & \leftarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ N & \leftarrow & 0. \end{array}} \xrightarrow{\left[\boxed{\begin{smallmatrix} 0 & 0 \\ 1 & 1 \end{smallmatrix}},\; \boxed{\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}}\right]} \boxed{\begin{array}{ccc} \mathcal{B} & \leftarrow & 0 \\ \downarrow{\scriptstyle \mathrm{id}} & & \downarrow \\ N & \xleftarrow{\mathrm{id}} & \mathcal{A}. \end{array}} \to 0$$

in $C(\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}})$ gives rise (cf. (3.12), (3.13), (3.14), and (3.10)) to the triangle

$$({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}})(N) \to ({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{A}})(\mathcal{A}) \oplus ({}^{\mathcal{E}}_{\mathcal{B}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}})(\mathcal{B}) \to \mathcal{E} \to [1]({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}})(N) \tag{3.15}$$

in $D(\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}})$.

If (E1) holds, the first two corners of the triangle (3.15) are compact, hence the third corner $\mathcal{E}$ is compact. This proves (E2).

Conversely assume that (E2) holds, i. e. the diagonal bimodule $\mathcal{E}$ is compact in $D(\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}})$.

We first prove that $\mathcal{B} \in \mathrm{per}(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$. Let $\mathcal{F} := \mathcal{A} \otimes_K \mathcal{B}^{\mathrm{op}} \subset \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ be the full dg K-subcategory (the upper right corner in the quiver picture (3.8)), and let $\mathcal{U}$ be its complement. Then there are no non-zero morphisms in $\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ from $\mathcal{F}$ to $\mathcal{U}$, hence we are in the situation of Section 3.2, cf. Remark 3.20. In particular we have the functor (cf. (3.7))

$$u^* := \mathrm{prod}_{\mathcal{U}}^{\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}} : D(\mathcal{U}) \to D(\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}})$$

and its right adjoint functor $u_* = \mathrm{res}_{\mathcal{U}}^{\mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}}$. Since the diagonal bimodule $\mathcal{E}$ (cf. (3.10)) "has support in $\mathcal{U}$" it satisfies $\mathcal{E} = u^* u_* \mathcal{E}$, and Lemma 3.23 shows that $u_* \mathcal{E} \in D(\mathcal{U})$ is compact.

Now consider the full dg K-subcategory $i : \mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}} \subset \mathcal{U}$ (the upper left corner in the quiver picture (3.8)). There are no non-zero morphisms from $\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}$ to its complement in $\mathcal{U}$, hence we are again in the situation of Section 3.2, and have the functor $i_* := \mathrm{res}^{\mathcal{U}}_{\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}}} : D(\mathcal{U}) \to D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$. Since $i_* u_* \mathcal{E} \in D(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$ is the diagonal bimodule $\mathcal{B}$ and compact by Lemma 3.22 we obtain $\mathcal{B} \in \mathrm{per}(\mathcal{B} \otimes_K \mathcal{B}^{\mathrm{op}})$.

Similarly, the inclusion $\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}} \hookrightarrow \mathcal{U}$ yields $\mathcal{A} \in \mathrm{per}(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}})$. Then in the triangle (3.15) the objects $({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{A}})(\mathcal{A}) \oplus ({}^{\mathcal{E}}_{\mathcal{B}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}})(\mathcal{B})$ and $\mathcal{E}$ are compact, hence $({}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}})(N)$ is compact. If we denote the inclusion $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}} \hookrightarrow \mathcal{E} \otimes_K \mathcal{E}^{\mathrm{op}}$ by v we have $v^* = {}^{\mathcal{E}}_{\mathcal{A}}\mathrm{prod}^{\mathcal{E}}_{\mathcal{B}}$. Hence compactness of $v^*(N)$ implies compactness of N in $D(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$ by Lemma 3.23, i. e. $N \in \mathrm{per}(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$. This proves (E1). $\square$

3.4. **Quillen adjunction for dg categories.** Let $R \to S$ be a morphism of graded commutative dg algebras. As in Section 2.7 we can consider the categories dgcat_R and dgcat_S of small dg R- and dg S-categories. There are obvious extension and restriction of scalars functors

$$\mathrm{dgcat}_R \underset{\mathrm{res}^S_R}{\overset{(? \otimes_R S)}{\rightleftarrows}} \mathrm{dgcat}_S$$

and we have an obvious adjunction $((? \otimes_R S), \mathrm{res}^S_R, \varphi)$. We sometimes write $\mathcal{A}_S := \mathcal{A} \otimes_R S$, if $\mathcal{A}$ is a dg R-category.

Proposition 3.28. *The adjunction $((? \otimes_R S), \mathrm{res}^S_R, \varphi)$ is a Quillen adjunction (where dgcat_R and dgcat_S are equipped with the model structure of Theorem 2.11).*

It is a Quillen equivalence if and only if $R \to S$ is a quasi-isomorphism.

Proof. From the proof of Theorem 2.11 one obtains explicitly a set I_R (resp. I_S) of generating cofibrations and a set J_R (resp. J_S) of generating trivial cofibrations for the model structure on dgcat_R (resp. dgcat_S). Then obviously the image of I_R (resp. J_R) under $(? \otimes_R S)$ is precisely I_S (resp. J_S), and the first statement follows from [Hov99a, Lemma 2.1.20].

To prove the second statement, assume that $((? \otimes_R S), \mathrm{res}^S_R, \varphi)$ is a Quillen equivalence. Note that $R \in \mathrm{dgcat}_R$ is (semi-free and) cofibrant and that any object of dgcat_S, for example S, is fibrant. Since $\mathrm{id} : R \otimes_R S \to S$ is a quasi-equivalence, the corresponding morphism $\varphi(\mathrm{id}) : R \to \mathrm{res}^S_R(S)$ is a quasi-equivalence. This just means that $R \to S$ is a quasi-isomorphism.

Conversely, assume that $R \to S$ is a quasi-isomorphism. Since any cofibrant dg R-category is R-h-flat (Lemma 2.14) and any dg S-category is fibrant, it is enough to show the following claim: Let $\mathcal{A}$ be an R-h-flat dg R-category and $\mathcal{B}$ a dg S-category. Then a morphism $f : \mathcal{A} \otimes_R S \to \mathcal{B}$ is a quasi-equivalence if and only if $\varphi(f) : \mathcal{A} \to \mathrm{res}^S_R(\mathcal{B})$ is a quasi-equivalence.

Given $f : \mathcal{A} \otimes_R S \to \mathcal{B}$, the morphism $\varphi(f)$ is the obvious composition

$$\mathcal{A} \to \mathrm{res}^S_R(\mathcal{A} \otimes_R S) \xrightarrow{\mathrm{res}^S_R(f)} \mathrm{res}^S_R(\mathcal{B}). \tag{3.16}$$

The first morphism of this composition can be viewed as the morphism $\mathcal{A} \otimes_R R \to \mathcal{A} \otimes_R S$ obtained from $R \to S$; it satisfies (qe2) for trivial reasons and (qe1) since $\mathcal{A}$ is R-h-flat. The second morphism is a quasi-equivalence if and only if f is a quasi-equivalence. Hence the 2-out-of-3-property proves our claim. □

3.5. **Smoothness and base change.** We continue the discussion in Section 3.4 and keep the assumptions there.

Let $Q : \mathrm{dgcat}_R \to \mathrm{dgcat}_R$ be a fixed cofibrant replacement functor. If $\mathcal{A}$ is a dg R-category, then $Q(\mathcal{A})$ is cofibrant and we have a trivial fibration $Q(\mathcal{A}) \to \mathcal{A}$ (= a cofibrant resolution) which is natural in $\mathcal{A}$.

The map $\mathcal{A} \mapsto Q(\mathcal{A})_S$ defines a functor $\mathrm{dgcat}_R \to \mathrm{dgcat}_S$. It maps weak equivalences to weak equivalences and hence induces the following functor between homotopy categories,

$$\begin{aligned} (? \otimes^L_R S) : \mathrm{Ho}(\mathrm{dgcat}_R) &\to \mathrm{Ho}(\mathrm{dgcat}_S), \\ \mathcal{A} &\mapsto \mathcal{A} \otimes^L_R S = Q(\mathcal{A})_S = Q(\mathcal{A}) \otimes_R S. \end{aligned}$$

Remark 3.29. *We explain how R-h-flatness may help when computing $\mathcal{A} \otimes^L_R S$. Let $\mathcal{A}' \to \mathcal{A}$ be an R-h-flat resolution, for example a cofibrant resolution (Lemma 2.14). Then $Q(\mathcal{A}) \to \mathcal{A}$ factors as a quasi-equivalence $Q(\mathcal{A}) \to \mathcal{A}'$ followed by $\mathcal{A}' \to \mathcal{A}$ (Lemma 3.1).*

Consider the commutative diagram

$$\begin{tikzcd} \mathcal{A} & \mathcal{A}' \arrow[l] \arrow[r] & \mathcal{A}'_S & = & \mathcal{A}' \otimes_R S \\ \mathcal{A} \arrow[u, equal] & Q(\mathcal{A}) \arrow[l] \arrow[r] \arrow[u] & Q(\mathcal{A})_S \arrow[u] & = & Q(\mathcal{A}) \otimes_R S \\ R \arrow[u] & R \arrow[l, equal] \arrow[r] \arrow[u] & S \arrow[u] \end{tikzcd}$$

whose lower three vertical arrows are merely symbolical. Note that $Q(\mathcal{A})_S \to \mathcal{A}'_S$ is a quasi-equivalence (use Lemma 2.9). Hence R-h-flat-resolutions are enough for computing the base change $Q(\mathcal{A})_S$ up to quasi-equivalence. Similarly, if S is R-h-flat, then we can replace $\mathcal{A}'$ by $\mathcal{A}$ in the above diagram (and $\mathcal{A}' \to \mathcal{A}$ by $\mathrm{id}_{\mathcal{A}}$) and obtain a quasi-equivalence $Q(\mathcal{A})_S \to \mathcal{A}_S$ (Lemma 2.15). These observations can be used for testing S-smoothness of $Q(\mathcal{A})_S$ (Lemma 3.12).

Theorem 3.30. *Let $R \to S$ be a morphism of graded commutative dg algebras and let $\mathcal{A}$ be a (small) dg R-category.*

(BC1) (Smoothness and base change) If $\mathcal{A}$ is R-smooth, then $Q(\mathcal{A})_S$ is S-smooth.

Now assume that $R \to S$ is a quasi-isomorphism.

(BC2) $\mathcal{A}$ is R-smooth if and only if $Q(\mathcal{A})_S$ is S-smooth.

(BC3) If $\mathcal{B}$ is a (small) dg S-category, then $\mathcal{B}$ is S-smooth if and only if $\mathrm{res}^S_R(\mathcal{B})$ is R-smooth.

Proof. We need some preparations. We abbreviate $\mathrm{res} := \mathrm{res}^S_R$.

Step 1: Let $\mathcal{T}$ be a dg R-category. The adjunction morphism

$$\mathcal{T} \to \mathrm{res}(\mathcal{T}_S) \tag{3.17}$$

is a morphism in dgcat_R and gives rise to the functor

$$\mathrm{prod}_{\mathcal{T}}^{\mathcal{T}_S} := \mathrm{prod}_{\mathcal{T}}^{\mathrm{res}\,\mathcal{T}_S} : C(\mathcal{T}) \to C(\mathrm{res}(\mathcal{T}_S)) = C(\mathcal{T}_S) \tag{3.18}$$

where the equality is the canonical identification explained in Remark 2.1. Explicitly, a dg $\mathcal{T}$-module M is mapped to the dg $\mathcal{T}_S$-module $M_S := \mathrm{prod}_{\mathcal{T}}^{\mathcal{T}_S}(M)$ which is given by

$$M_S(T) = M(T) \otimes_R S$$

at $T \in \mathcal{T}_S$ and has the obvious action morphisms. On the level of derived categories we obtain the functor

$$L\,\mathrm{prod}_{\mathcal{T}}^{\mathcal{T}_S} : D(\mathcal{T}) \to D(\mathrm{res}\,\mathcal{T}_S) = D(\mathcal{T}_S), \tag{3.19}$$

mapping M to $p(M)_S$, which preserves compact objects (as explained in Section 2.6.4).

Step 2: Let $\mathcal{A}$, $\mathcal{B}$ be dg R-categories. Then (3.18) applied to $\mathcal{T} = \mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}}$ yields a functor

$$\mathrm{prod}_{\mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}}}^{\mathcal{A}_S \otimes_S (\mathcal{B}_S)^{\mathrm{op}}} : C(\mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}}) \to C(\mathcal{A}_S \otimes_S (\mathcal{B}_S)^{\mathrm{op}}).$$

Explicitly, let X be a dg $\mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}}$ module. Then X_S is given on $(A, B) \in \mathcal{A}_S \otimes_S (\mathcal{B}_S)^{\mathrm{op}}$ by

$$X_S(A, B) = X(A, B) \otimes_R S$$

with obvious action morphisms. In particular, for $\mathcal{A} = \mathcal{B}$ this shows that the diagonal bimodule $\mathcal{A}$ is mapped to the diagonal bimodule $\mathcal{A}_S$. We need a similar statement on the level of derived categories.

Step 3: Let $\mathcal{A}$, $\mathcal{B}$ and X be as above, but assume in addition that $\mathcal{A}$ and $\mathcal{B}$ have cofibrant morphism spaces and that $X(A, B)$ is R-h-flat for all $A \in \mathcal{A}$, $B \in \mathcal{B}^{\mathrm{op}}$. Let $\gamma : p(X) \to X$ be a cofibrant resolution of X in $C(\mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}})$. We claim that $p(X)_S \to X_S$ is a quasi-isomorphism. By Proposition 2.10 all $p(X)(A, B)$ are cofibrant dg R-modules and in particular R-h-flat. Hence, by Lemma 2.9,

$$(p(X)(A, B)) \otimes_R S \xrightarrow{\gamma(A,B) \otimes_R \mathrm{id}_S} X(A, B) \otimes_R S$$

is a quasi-isomorphism for all $A \in \mathcal{A}$, $B \in \mathcal{B}$, proving our claim. Hence

$$L\,\mathrm{prod}_{\mathcal{A} \otimes_R \mathcal{B}^{\mathrm{op}}}^{\mathcal{A}_S \otimes_S (\mathcal{B}_S)^{\mathrm{op}}}(X) = p(X)_S \xrightarrow{\sim} X_S$$

in $D(\mathcal{A}_S \otimes_S (\mathcal{B}_S)^{\mathrm{op}})$.

Step 4: Assume that $\widetilde{\mathcal{A}}$ is a cofibrant dg R-category. Then $\widetilde{\mathcal{A}}$ has cofibrant and K-h-flat morphism spaces (Lemma 2.14), so we can apply Step 3 to the diagonal bimodule $X := \widetilde{\mathcal{A}}$. This implies that

$$L\,\mathrm{prod}_{\widetilde{\mathcal{A}} \otimes_R \widetilde{\mathcal{A}}^{\mathrm{op}}}^{\widetilde{\mathcal{A}}_S \otimes_S (\widetilde{\mathcal{A}}_S)^{\mathrm{op}}} : D(\widetilde{\mathcal{A}} \otimes_R \widetilde{\mathcal{A}}^{\mathrm{op}}) \to D(\widetilde{\mathcal{A}}_S \otimes_S (\widetilde{\mathcal{A}}_S)^{\mathrm{op}}) \tag{3.20}$$

maps the diagonal bimodule $\widetilde{\mathcal{A}}$ to an object isomorphic to the diagonal bimodule $\widetilde{\mathcal{A}}_S$.

Now we can prove our claims.

(BC1): Assume that $\mathcal{A}$ is R-smooth. Then by definition the diagonal bimodule $Q(\mathcal{A})$ is compact. Hence from Step 4 with $\widetilde{\mathcal{A}} = Q(\mathcal{A})$ (and the fact that the functor in (3.20) preserves compact objects) we see that the diagonal bimodule $Q(\mathcal{A})_S$ is compact. This is equivalent to $Q(\mathcal{A})_S$ being S-smooth (note that $Q(\mathcal{A})_S$ is a cofibrant dg S-category, as follows from Proposition 3.28 and the fact that any left Quillen functor preserves cofibrant objects).

Assume now that $R \to S$ is a quasi-isomorphism.

(BC2): If $\mathcal{T}$ is an R-h-flat dg R-category, (3.17) is a quasi-equivalence (cf. the explanation below (3.16)) and (3.19) is an equivalence (cf. before (2.13)). The cofibrant dg R-category $Q(\mathcal{A})$ is R-h-flat (Lemma 2.14). But then also $Q(\mathcal{A}) \otimes_R Q(\mathcal{A})$ is R-h-flat,

and hence (3.20) for $\widetilde{\mathcal{A}} = Q(\mathcal{A})$ is an equivalence mapping the diagonal bimodule $Q(\mathcal{A})$ to (an object isomorphic to) the diagonal bimodule $Q(\mathcal{A})_S$. This means that $\mathcal{A}$ is smooth if and only if $Q(\mathcal{A})_S$ is smooth.

(BC3): Let $\mathcal{B}$ be a dg S category and consider the cofibrant resolution $Q(\operatorname{res}(\mathcal{B})) \to \operatorname{res}(\mathcal{B})$. Then $Q(\operatorname{res}(\mathcal{B}))_S \to \mathcal{B}$ is a quasi-equivalence, since $((? \otimes_R S), \operatorname{res}^S_R, \varphi)$ is a Quillen equivalence (Prop. 3.28). (It is even a cofibrant resolution; use (trFib1) and (trFib2).) Now use Lemma 3.12 and (BC2). $\square$

3.6. **Locally perfect categories and smoothness.** We extend some presumably well-known results (cf. e. g. part of the proof of [Kel08, Lemma 4.1], or [Shk07, Prop. 3.4]) to the dg K-setting. This Section may be skipped: only Corollary 3.37 is used later on in one of the two proofs of Proposition 3.40.

Definition 3.31 (cf. [TV07, Def. 2.4])**.** *A dg K-category $\mathcal{A}$ is **locally K-perfect** (or **locally K-proper**) if $\mathcal{A}(A, A')$ is a compact dg K-module (i. e. in $\operatorname{per}(K)$) for all $A, A' \in \mathcal{A}$.*

It is easy to show that local K-perfectness is invariant under quasi-equivalences.

Definition 3.32. *Let $\mathcal{A}$ be dg K-category. A dg $\mathcal{A}$-module M is called **locally K-perfect** (or **locally K-proper**) if $M(A)$ is compact when considered as an object of $D(K)$, for all $A \in \mathcal{A}$. The full subcategory of $D(\mathcal{A})$ consisting of locally K-perfect dg $\mathcal{A}$-modules is denoted by $D_{lp}(\mathcal{A})$.*

Clearly, local K-perfectness of dg $\mathcal{A}$-modules is invariant under isomorphisms in $D(\mathcal{A})$.

Lemma 3.33. *Let $\mathcal{A}$ be a dg K-category. Then $\mathcal{A}$ is locally K-perfect if and only if* $\operatorname{per}(\mathcal{A}) \subset D_{lp}(\mathcal{A})$.

Proof. In general, $D_{lp}(\mathcal{A})$ is a strict full triangulated subcategory of $D(\mathcal{A})$ and closed under summands. Hence it contains $\operatorname{per}(\mathcal{A})$ if and only if it contains all $\widehat{A}$, for $A \in \mathcal{A}$; this condition just means that $\mathcal{A}$ is locally K-perfect. $\square$

Lemma 3.34. *If $F : \mathcal{B} \to \mathcal{A}$ is a quasi-equivalence of dg K-categories, then restriction along F induces an equivalence $\operatorname{res}^{\mathcal{A}}_{\mathcal{B}} : D_{lp}(\mathcal{A}) \to D_{lp}(\mathcal{B})$.*

Proof. We know from Section 2.6.4 that $\operatorname{res}^{\mathcal{A}}_{\mathcal{B}} : D(\mathcal{A}) \xrightarrow{\sim} D(\mathcal{B})$ is an equivalence. If M is a locally K-perfect dg $\mathcal{A}$-module, then obviously $\operatorname{res}^{\mathcal{A}}_{\mathcal{B}}(M)$ is locally K-perfect. Since local K-perfectness is invariant under isomorphisms in $D(\mathcal{B})$ it is enough to show that the converse is also true.

Let $A \in \mathcal{A}$. Then there is an object $B \in \mathcal{B}$ such that A and $F(B)$ are isomorphic in $[\mathcal{A}]$. Application of $[M] : [\mathcal{A}^{\mathrm{op}}] \to \mathcal{H}(\mathcal{A})$ shows that $M(A)$ and $M(F(B))$ are isomorphic

in $\mathcal{H}(\mathcal{A})$ and a fortiori in $D(\mathcal{A})$. Hence if $\operatorname{res}^{\mathcal{A}}_{\mathcal{B}}(M)$ is locally K-perfect, then $M(F(B))$ and hence $M(A)$ are compact in $D(K)$. This implies that M is locally K-perfect. □

Proposition 3.35. *Let $\mathcal{A}$ be a dg K-category. If $\mathcal{A}$ is K-smooth, then $D_{lp}(\mathcal{A}) \subset \operatorname{per}(\mathcal{A})$.*

Proof. By Section 2.6.4 and Lemma 3.34 we know that restriction along a K-h-flat resolution $\widetilde{\mathcal{A}} \to \mathcal{A}$ induces an equivalence $\operatorname{res}^{\mathcal{A}}_{\widetilde{\mathcal{A}}} : D(\mathcal{A}) \to D(\widetilde{\mathcal{A}})$ identifying $\operatorname{per}(\mathcal{A})$ with $\operatorname{per}(\widetilde{\mathcal{A}})$ and $D_{lp}(\mathcal{A})$ with $D_{lp}(\widetilde{\mathcal{A}})$. Hence it is enough to prove the claim under the additional assumption that $\mathcal{A}$ is K-h-flat.

We start with some preparations. If A is an object of a dg K-category $\mathcal{A}$, there is a unique dg K-functor $K \to \mathcal{A}$ (where K is viewed as a dg K-algebra) mapping the unique object of K to A. We denote this functor by $K = K_A \to \mathcal{A}$ to indicate its dependence on A.

Let $\mathcal{A}$, $\mathcal{B}$ be dg K-categories and let M be a dg $\mathcal{A}$-module. Consider the dg K-functor

$$\begin{aligned} F_M : \mathcal{M}od(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}) &\to \mathcal{M}od(\mathcal{B}), \\ X &\mapsto M \otimes_{\mathcal{A}} X, \end{aligned}$$

and let $LF_M : D(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}) \to D(\mathcal{B})$, $X \mapsto M \otimes_{\mathcal{A}} p(X)$, be its left derived functor.

If X is a cofibrant dg $\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}$-module, then $p(X) \to X$ is a quasi-isomorphism between cofibrant objects in $C(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$ and an isomorphism in $\mathcal{H}(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}})$, hence $LF_M(X) \xrightarrow{\sim} F_M(X)$ in $D(\mathcal{B})$.

Let $B \in \mathcal{B}$, $A \in \mathcal{A}^{\mathrm{op}}$. It is easy to see that

$$F_M(\widehat{(B,A)}) = (M(A)) \otimes_K \widehat{B} = (\operatorname{res}^{\mathcal{A}}_{K_A}(M)) \otimes_K \widehat{B}$$

in $C(\mathcal{B})$. Since $\widehat{(B,A)}$ is cofibrant we have

$$LF_M(\widehat{(B,A)}) \xrightarrow{\sim} (\operatorname{res}^{\mathcal{A}}_{K_A}(M)) \otimes_K \widehat{B} \tag{3.21}$$

in $D(\mathcal{B})$.

Let $\operatorname{prod}^{\mathcal{B}}_{K_B} : \mathcal{M}od(K) \to \mathcal{M}od(\mathcal{B})$ be the extension of scalars functor along $K = K_B \to \mathcal{B}$. Obviously $\operatorname{prod}^{\mathcal{B}}_{K_B}(N) = N \otimes_K \widehat{B}$. Assume now that $\mathcal{B}$ is K-h-flat. Then $\operatorname{prod}^{\mathcal{B}}_{K_B}$ preserves quasi-isomorphism. Hence $p(\operatorname{res}^{\mathcal{A}}_{K_A}(M)) \to \operatorname{res}^{\mathcal{A}}_{K_A}(M)$ yields an isomorphism

$$L\operatorname{prod}^{\mathcal{B}}_{K_B}(\operatorname{res}^{\mathcal{A}}_{K_A}(M)) \xrightarrow{\sim} (\operatorname{res}^{\mathcal{A}}_{K_A}(M)) \otimes_K \widehat{B} \tag{3.22}$$

in $D(\mathcal{B})$.

Assume in addition that $M \in D_{lp}(\mathcal{A})$. Since $L\operatorname{prod}^{\mathcal{B}}_{K_B}$ preserves compact objects, (3.21) and (3.22) show that $LF_M(\widehat{(B,A)}) \in \operatorname{per}(\mathcal{B})$.

This implies that LF_M induces a functor

$$LF_M : \operatorname{per}(\mathcal{B} \otimes_K \mathcal{A}^{\mathrm{op}}) \to \operatorname{per}(\mathcal{B})$$

(under the assumptions that $\mathcal{B}$ is K-h-flat and M is locally K-perfect).

Now assume that $\mathcal{A}$ is K-h-flat and K-smooth. Then $\mathcal{A} \in \operatorname{per}(\mathcal{A} \otimes_K \mathcal{A}^{\mathrm{op}})$ by Lemma 3.6. The above arguments applied to $\mathcal{B} = \mathcal{A}$ show: If M is a locally K-perfect dg $\mathcal{A}$-module, then

$$LF_M(\mathcal{A}) \in \operatorname{per}(\mathcal{A}).$$

We now prove $D_{lp}(\mathcal{A}) \subset \operatorname{per}(\mathcal{A})$. Let $N \in D_{lp}(\mathcal{A})$. Then $p(N)$ is locally K-perfect and we obtain

$$\operatorname{per}(\mathcal{A}) \ni LF_{p(N)}(\mathcal{A}) = p(N) \otimes_{\mathcal{A}} p(\mathcal{A})$$

Since $p(\mathcal{A}) \to \mathcal{A}$ is a quasi-isomorphism and $p(N)$ is $\mathcal{A}$-h-flat (Lemma 2.8) we obtain

$$p(N) \otimes_{\mathcal{A}} p(\mathcal{A}) \xrightarrow{\sim} p(N) \otimes_{\mathcal{A}} \mathcal{A} = p(N) \xrightarrow{\sim} N$$

in $D(\mathcal{A})$. This shows $N \in \operatorname{per}(\mathcal{A})$. □

Corollary 3.36. *Let $\mathcal{A}$ be a dg K-category that is K-smooth and locally K-perfect. Then*

$$D_{lp}(\mathcal{A}) = \operatorname{per}(\mathcal{A}).$$

Proof. Follows from Proposition 3.35 and Lemma 3.33. □

Corollary 3.37 (cf. [Shk07, Prop. 3.4])**.** *Let A be a dg k-algebra over a field k, and let M be a dg A-module.*

(a) If A is k-smooth, then $\dim_{\mathsf{k}} H(M) < \infty$ implies $M \in \operatorname{per}(A)$.
(b) If A is k-smooth and $H(A)$ is finite dimensional, then $\dim_{\mathsf{k}} H(M) < \infty$ if and only if $M \in \operatorname{per}(A)$.

Proof. Obviously, $M \in D_{lp}(A)$ if and only if $M \in \operatorname{per}(\mathsf{k})$ if and only if $H(M)$ is finite dimensional. Similarly, A is locally perfect over k if and only if $H(A)$ is finite dimensional. Now use Proposition 3.35 and Corollary 3.36. □

3.7. **Smoothness criteria.** In this Section k will be a field (viewed as a dg ring concentrated in degree zero). Our aim is to prove the two smoothness criteria provided by Propositions 3.40 and 3.43 below. The latter Proposition will be essential for Section 4.

We need some preparations for the proof of the first criterion.

Lemma 3.38 ([ELO09, Lemma 9.5])**.** *Let k be a field and A a dg k-algebra. Assume that $H^0(A) = \mathsf{k}$ and $H^i(A) = 0$ for all $i < 0$. Then there is a dg k-subalgebra U of A such that $U^0 = \mathsf{k}$, $U^i = 0$ for all $i < 0$, and the inclusion $U \hookrightarrow A$ is a quasi-isomorphism.*

Proof. Let $C \subset Z^1(A)$ be a linear subspace such that the restriction of $Z^1(A) \to H^1(A)$ to C is an isomorphism, and let $B \subset A^1$ be a linear subspace such that $d : A^1 \to A^2$ induces an isomorphism $B \xrightarrow{\sim} d(A^1)$. Define $U^i := 0$ for $i < 0$, $U^0 := \mathsf{k}$, $U^1 := C \oplus B$, $U^i := A^i$ for $i > 1$, and take $U := \bigoplus U^i$. □

If $N = \bigoplus_{i \in \mathbb{Z}} N^i$ is a graded abelian group which is bounded and nonzero, we define its amplitude by

$$\operatorname{ampl}(N) := \max\{i \in \mathbb{Z} \mid N^i \neq 0\} - \min\{i \in \mathbb{Z} \mid N^i \neq 0\}.$$

Lemma 3.39. *Let k be a field and let A be a dg k-algebra such that $H^i(A) = 0$ for all $i < 0$ and $H^0(A) = \mathsf{k}$. Assume that $H(A)$ is bounded above and let $m \in \mathbb{N}$ be maximal such that $H^m(A) \neq 0$; assume that $m > 0$ (i. e. $\mathsf{k} \neq H(A)$). Let $M \in \operatorname{per}(A)$ with $M \ncong 0$. Then $H(M)$ is bounded and nonzero and*

$$\operatorname{ampl}(H(M)) \geq m.$$

Moreover, $\dim_{\mathsf{k}} H^{\mathrm{top}}(M) \geq \dim_{\mathsf{k}} H^m(A)$, where $H^{\mathrm{top}}(M)$ is the highest non-vanishing cohomology of M. In particular, M has at least two nonvanishing cohomology groups.

Proof. Let $U \subset A$ be as in Lemma 3.38. Then res^A_U induces an equivalence $\operatorname{per}(A) \xrightarrow{\sim} \operatorname{per}(U)$ preserving cohomology. Hence by replacing A by U we can assume in addition that $A^i = 0$ for all $i < 0$ and that $A^0 = \mathsf{k}$.

Let $M \in \operatorname{per}(A)$, $M \ncong 0$. Since A is positively graded with $A^0 = \mathsf{k} \subset A$ as a dg subalgebra, we can assume that M has a finite filtration $0 = F_0 \subset F_1 \subset \cdots \subset F_n = M$ in $C(A)$ such that $F_i/F_{i-1} \cong [l_i]A$ for suitable $l_1 \geq l_2 \geq \cdots \geq l_n$ (this follows from [Sch11, Thm. 1]; see Rem. 9 there for a picture).

Obviously $M^i = 0$ for $i < -l_1$, and $1 \in A^0 = ([l_1]A)^{-l_1} \cong {F_1}^{-l_1} \subset M^{-l_1}$ is a cocycle in M that defines an nonzero element of $H^{-l_1}(M)$.

We prove by induction on the length $n \geq 1$ of the filtration that there is a surjection $H^{-l_n+m}(M) \to H^m(A) \neq 0$ and that $H^{-l_n+m+i}(M)$ vanishes for $i > 0$. For $n = 1$ this is obviously true. Let $n \geq 2$ and assume that the claim is true for $n-1$. Consider the short exact sequence

$$0 \to F_{n-1} \hookrightarrow M \twoheadrightarrow [l_n]A \to 0.$$

For $i > 0$ we have obviously $H^{-l_n+m+i}([l_n]A) = H^{m+i}(A) = 0$ and by induction $H^{-l_n+m+i}(F_{n-1}) = H^{-l_{n-1}+m+(i+l_{n-1}-l_n)}(F_{n-1}) = 0$ since $l_{n-1} - l_n \geq 0$. The long exact cohomology sequence then proves that $H^{-l_n+m+i}(M) = 0$ for $i > 0$ and that

$$H^{-l_n+m}(M) \to H^{-l_n+m}([l_n]A) = H^m(A) \neq 0$$

is surjective.

We have proved that the lowest (resp. highest) cohomology of M lives in degree $-l_1$ (resp. $-l_n + m$). Hence $\operatorname{ampl}(H(M)) = -l_n + m + l_1 \geq m$. □

Proposition 3.40. *Let* k *be a field and let* A *be a dg* k*-algebra such that* $H^i(A) = 0$ *for all* $i < 0$, $H^0(A) = \mathsf{k}$, *and* $H(A)$ *is bounded above. Then* A *is* k*-smooth if and only if* $\mathsf{k} = H(A)$.

Proof. Recall that smoothness is invariant under quasi-isomorphisms of dg k-algebras (Lemma 3.12). If $\mathsf{k} = H(A)$ then $\mathsf{k} \to A$ is a quasi-isomorphism and hence A is k-smooth.

Assume that $\mathsf{k} \neq H(A)$. We give two proofs showing that A is not k-smooth.

First proof: Lemma 3.38 shows that we can assume in addition that $A^i = 0$ for all $i < 0$ and that $A^0 = \mathsf{k}$. Then it is obvious that A has a one dimensional "augmentation module" k. If A is k-smooth, part (a) of Corollary 3.37 implies that $\mathsf{k} \in \operatorname{per}(A)$. This contradicts Lemma 3.39.

Second (easier) proof: Since we work over a field, smoothness of A is equivalent to A being in $\operatorname{per}(A \otimes_{\mathsf{k}} A^{\mathrm{op}})$. Let $r \in \mathbb{Z}$ be maximal such that $H^r(A) \neq 0$. By assumption $0 < r < \infty$. Since $H(A \otimes_{\mathsf{k}} A^{\mathrm{op}}) \cong H(A) \otimes_{\mathsf{k}} H(A^{\mathrm{op}})$ (at least as graded k-modules), $A \otimes_{\mathsf{k}} A^{\mathrm{op}}$ satisfies the assumptions of Lemma 3.39, and $m = 2r$ is the degree of the highest nonzero cohomology group of $A \otimes_{\mathsf{k}} A^{\mathrm{op}}$. The assumption $A \in \operatorname{per}(A \otimes_{\mathsf{k}} A^{\mathrm{op}})$ yields the contradiction $r = \operatorname{ampl}(H(A)) \geq 2r$. Hence A is not k-smooth. □

Before we can give the proof of the second criterion, Proposition 3.43, we explain some preparatory results and introduce some notation. Let

$$M = (\ldots \to M_i \xrightarrow{f_i} M_{i+1} \to \ldots)$$

be a complex in $C(\mathsf{k})$, i. e. a complex of complexes in k-vector spaces. We associate with M a double complex $\operatorname{Dbl}(M)$ (of k-vector spaces) defined as follows:

$$\begin{aligned}
\operatorname{Dbl}(M)^{ij} &:= M_i^j, & \\
d' : \operatorname{Dbl}(M)^{ij} &\to \operatorname{Dbl}(M)^{i+1,j}, & m &\mapsto f_i(m), \\
d'' : \operatorname{Dbl}(M)^{ij} &\to \operatorname{Dbl}(M)^{i,j+1}, & m &\mapsto (-1)^i d_{M_i}(m).
\end{aligned}$$

Let (N, d', d'') be a double complex. For $l \in \mathbb{Z}$ let $F^l N$ be the subcomplex such that $(F^l N)^{ij} = 0$ if $i < l$ and $(F^l N)^{ij} = N^{ij}$ if $i \geq l$. The $F^l N$ define a decreasing filtration $\cdots \supset F^l N \supset F^{l+1} N \supset \ldots$ on N. We define the total complex $\operatorname{Tot}(N)$ associated to N,

a complex of vector spaces, by

$$\begin{aligned}
\operatorname{Tot}(N)^n &:= \bigoplus_{i+j=n} N^{ij}, \\
d : \operatorname{Tot}(N)^n &\to \operatorname{Tot}(N)^{n+1}, \\
N^{ij} \ni m &\mapsto d'(m) + d''(m) \in N^{i+1,j} \oplus N^{i,j+1} \subset \operatorname{Tot}(N)^{n+1}.
\end{aligned}$$

It has an induced filtration. We define $\operatorname{tot} := \operatorname{Tot} \circ \operatorname{Dbl}$.

Let K be a dg k-algebra. Given a complex M in $C(K)$, i.e. a complex of dg K-modules, then each "column" $\operatorname{Dbl}(M)^{i*}$ of $\operatorname{Dbl}(M)$ is obviously a graded K-module (and differential and K-module structure are related by $d''(mk) = d''(m)k + (-1)^{i+j} m d_K(k)$ for $m \in M_i^j$ and $k \in K$) and $d' : \operatorname{Dbl}(M)^{i*} \to \operatorname{Dbl}(M)^{i+1,*}$ is K-linear. It follows that $\operatorname{tot}(M)$ becomes a dg K-module which is equipped with a decreasing filtration by dg K-submodules $F^l \operatorname{tot}(M)$. Moreover it is clear that Dbl and Tot are functorial.

Lemma 3.41. *Let k be a field and K a graded k-algebra such that $K^i = 0$ for $i < 0$, $K^0 = \mathsf{k}$. Let M be a graded (right) K-module which is bounded below, i.e. there is $m \in \mathbb{Z}$ such that $M^i = 0$ for all $i < m$. Then M has a "minimal" graded free resolution, i.e. there is a complex*

$$P = (\ldots \to P_i \xrightarrow{p_i} P_{i+1} \to \ldots \to P_{-1} \xrightarrow{p_{-1}} P_0 \to 0 \to \ldots)$$

of graded free K-modules together with a quasi-isomorphism $P \to M$, given by $p_0 : P_0 \to M$ such that

- *(a) the obvious morphism $\operatorname{tot}(P) \to M$ (induced by p_0) is a cofibrant resolution of M in $C(K)$ and is bounded below by m, i.e. $\operatorname{tot}(P)^i = 0$ for $i < m$ (here we view $M = (M, d_M = 0)$ and all $(P_i, d_{P_i} = 0)$ as dg K-modules, where $K = (K, d_K = 0)$);*
- *(b) the differential in the complex $P \otimes_K \mathsf{k}$ in $C(\mathsf{k})$ vanishes (and in particular the differential in $\operatorname{tot}(P) \otimes_K \mathsf{k} = \operatorname{tot}(P \otimes_K \mathsf{k})$ vanishes);*
- *(c) if K is a (right) Noetherian ring and M is a finitely generated K-module, then all P_i are finitely generated K-modules;*
- *(d) if M has finite projective dimension s as a K-module, then $P_i = 0$ for all $i < -s$.*

In particular, if K is Noetherian and M is finitely generated and of finite projective dimension, then $\dim_{\mathsf{k}} H(M \otimes_K^L \mathsf{k}) < \infty$, where $M \otimes_K^L \mathsf{k} \in D(\mathsf{k})$ is obtained by extension of scalars along the obvious (augmentation) morphism $K \to \mathsf{k}$ of dg k-algebras.

Proof. We sometimes write $\overline{N} = N \otimes_K \mathsf{k}$ if N is a graded K-module, and similarly for morphisms.

Since $P_0 := \overline{M} \otimes_{\mathsf{k}} K$ is a graded projective K-module, the obvious morphism $P_0 \to \overline{M}$ factors through $M \to \overline{M}$ to a morphism $p_0 : P_0 \to M$. Note that $\overline{p_0} : \overline{P_0} \to \overline{M}$ is an isomorphism; this implies in particular that $\overline{\operatorname{cok} p_0} = 0$, hence $\operatorname{cok} p_0 = 0$ (since $\operatorname{cok} p_0$ is bounded below), and p_0 is surjective. Apply this method now to the kernel of p_0. By induction we obtain a graded projective resolution

$$\ldots \to P_i \xrightarrow{p_i} P_{i+1} \to \ldots \to P_{-1} \xrightarrow{p_{-1}} P_0 \xrightarrow{p_0} M \to 0$$

of M, such that all $\overline{p_i}$ for $i \leq -1$ vanish. This shows (b). Since the "vertical" differential d'' in $\operatorname{Tot}(P)$ vanishes it is obvious that $\operatorname{tot}(P) \to M$ is a surjective quasi-isomorphism. There is an obvious filtration on $\operatorname{tot}(P)$ showing that $\operatorname{tot}(P)$ is semi-free as a dg K-module and and hence cofibrant. By construction P_0 is generated in degrees $\geq m$ and $p_0 : P_0 \to M$ is an isomorphism in degrees $\leq m$; hence its kernel is generated in degrees $\geq m+1$. So P_{-1} is generated in degrees $\geq m+1$ and p_{-1} induces an isomorphism onto the kernel of p_0 in degrees $\leq m+1$. By induction P_i is generated in degrees $\geq m-i$. This implies that $\operatorname{tot}(P)$ is generated in degrees $\geq m$. These arguments show (a). Claim (c) is obvious.

Assume that M has projective dimension s. Then p_{-s} induces an isomorphism from P_{-s} onto the kernel of p_{-s+1} (see [Bou07, X.§8.7, Prop. 8 (and Cor. 2)]). Hence $P_i = 0$ for all $i < -s$, proving (d).

Assume that K is Noetherian and M is finitely generated and of finite projective dimension. Then $M \otimes^L_K \mathsf{k}$ is isomorphic to the dg k-module $\operatorname{tot}(P) \otimes_K \mathsf{k} = \operatorname{tot}(P \otimes_K \mathsf{k})$ with vanishing differential and finite (total) dimension. $\square$

Lemma 3.42. *Let* k *be a field and* K *a graded* k*-algebra such that* $K^i = 0$ *for* $i < 0$, $K^0 = \mathsf{k}$. *We view* K *as a dg (*k*-)algebra with differential* $d_K = 0$.

Let M *be a dg* K*-module that (or whose cohomology) is concentrated in degrees* $\geq m$ *for some* $m \in \mathbb{Z}$. *Then* $H(M \otimes^L_K \mathsf{k})$ *is concentrated in degrees* $\geq m$, *and* $M \otimes^L_K \mathsf{k}$ *is acyclic if and only if* M *is acyclic.*

Proof. If M is acyclic it is clear that $M \otimes^L_K \mathsf{k}$ is acyclic.

If the cohomology of M is concentrated in degrees $\geq m$, replace M by the dg K-submodule defined as follows (cf. Lemma 3.38): It is zero in all degrees $< m$, coincides with M in all degrees $> m$, and in degree m it is the direct sum of a subspace of $Z(M)^m$ that goes isomorphically onto $H^m(M)$ and a subspace of M^m that goes isomorphically onto $B^{m+1}(M)$.

Hence we can assume without loss of generality that M is concentrated in degrees $\geq m$. Since $d_K = 0$ the morphism $d_M : M \to [1]M$ is (K-linear and hence) a morphism in $C(K)$. The short exact sequence $Z(M) \hookrightarrow M \overset{d_M}{\twoheadrightarrow} [1]B(M)$ in $C(K)$ yields a triangle in

$D(K)$ and by rotation a triangle

$$B(M) \to Z(M) \to M \xrightarrow{d_M} [1]B(M)$$

for some morphism $B(M) \to Z(M)$ in $D(K)$. Since $B(M)$ and $Z(M)$ have vanishing differential and are concentrated in degrees $\geq m+1$ and $\geq m$ respectively, Lemma 3.41 yields cofibrant resolutions $P \to B(M)$ and $Q \to Z(M)$ in $C(K)$ such that P and Q are graded free as K-modules, concentrated in degrees $\geq m+1$ and $\geq m$ respectively, and such that the differential of $P \otimes_K \mathsf{k}$ and of $Q \otimes_K \mathsf{k}$ vanishes. Replacing the first two terms of the above triangle by these cofibrant resolutions yields a triangle

$$P \to Q \to M \to [1]P.$$

Since P is h-projective (even cofibrant) we can assume that the morphism $P \to Q$ is represented by a morphism $e : P \to Q$ in $C(K)$. Note that $\mathrm{Cone}(e)$ is h-projective and concentrated in degrees $\geq m$. Hence there is a quasi-isomorphism $\mathrm{Cone}(e) \to M$. This implies that $\mathrm{Cone}(e \otimes_K \mathrm{id}_{\mathsf{k}}) = \mathrm{Cone}(e) \otimes_K \mathsf{k} \cong M \otimes^L_K \mathsf{k}$ has cohomology concentrated in degrees $\geq m$.

If $M \otimes^L_K \mathsf{k}$ is acyclic, then $e \otimes_K \mathrm{id}_{\mathsf{k}} : P \otimes_K \mathsf{k} \to Q \otimes_K \mathsf{k}$ is a quasi-isomorphism, and an isomorphism since the differentials of $P \otimes_K \mathsf{k}$ and $Q \otimes_K \mathsf{k}$ vanish. A morphism f of bounded below graded free K-modules is an isomorphism if and only if $f \otimes_K \mathrm{id}_{\mathsf{k}}$ is an isomorphism. This implies that $e : P \to Q$ is an isomorphism, and hence $\mathrm{Cone}(e) \cong M$ is acyclic. □

Proposition 3.43. *Let k be a field and K a graded commutative graded k-algebra such that $K^i = 0$ for $i < 0$, $K^0 = \mathsf{k}$. We view K as a (graded commutative) dg k-algebra with differential $d_K = 0$. Assume that K is a (right) Noetherian ring of finite global dimension. Let A be a dg K-algebra such that $H(A)$ is a finitely generated $H(K)$-module (of course $H(K) = K$) satisfying $H(A)^i = 0$ for $i < 0$ and $H^0(A) = \mathsf{k}$. Furthermore we assume that the structure morphism $K \to A$ induces a monomorphism $K^1 = H^1(K) \hookrightarrow H^1(A)$ on the first cohomology groups (this is the case for example if K^1 vanishes).*

Then A is K-smooth if and only if (the structure morphism) $K \to A$ is a quasi-isomorphism.

Proof. If $K \to A$ is a quasi-isomorphism then A is obviously K-smooth (Lemma 3.12).

Consider $K \to A$ as a morphism of dg K-modules, and fit it into a triangle

$$K \to A \to Q \to [1]K$$

in $D(K)$. Since $K \to A$ induces isomorphisms on cohomology in all degrees ≤ 0 and a monomorphism in degree one, $H(Q)$ is concentrated in degrees ≥ 1. Extension of scalars

along $K \twoheadrightarrow K/K^{>0} = \mathsf{k}$ yields the triangle

$$\mathsf{k} \to A \otimes_K^L \mathsf{k} \to Q \otimes_K^L \mathsf{k} \to [1]\mathsf{k}$$

Lemma 3.42 and the long exact cohomology sequence show that $H^i(A \otimes_K^L \mathsf{k})$ vanishes for $i < 0$ and that $\mathsf{k} \xrightarrow{\sim} H^0(A \otimes_K^L \mathsf{k})$ canonically; moreover $\mathsf{k} \xrightarrow{\sim} H(A \otimes_K^L \mathsf{k})$ if and only if $K \to A$ is a quasi-isomorphism.

The dg K-version of [Dri04, Lemma 13.5] and Lemma 2.12 yield a cofibrant resolution $\widetilde{A} \to A$ where $\widetilde{A}$ is a (semi-free and) cofibrant dg K-algebra $\widetilde{A}$. Let $B := \widetilde{A} \otimes_K \mathsf{k}$.

Since $\widetilde{A} \to A$ can be viewed as a cofibrant resolution in $C(K)$ by Lemma 2.14, we obtain $B \cong A \otimes_K^L \mathsf{k}$ in $D(\mathsf{k})$. From the above we obtain $H(B)^i = 0$ for $i < 0$ and $H^0(B) = \mathsf{k}$; moreover $\mathsf{k} = H(B)$ if and only if $K \to A$ is a quasi-isomorphism.

Claim: $H(B)$ is bounded above (even finite dimensional as a k-vector space).

Assuming this claim, we proceed as follows. Let A be K-smooth. Then $Q(A) \otimes_K \mathsf{k}$ is k-smooth by Theorem 3.30, part (BC1), and the same is true for B by Remark 3.29. Now Proposition 3.40 shows that $\mathsf{k} = H(B)$. Hence $K \to A$ is a quasi-isomorphism.

Proof of the claim: We prove that $H(A \otimes_K^L \mathsf{k})$ has finite k-dimension. Let P be a "minimal" graded free resolution of the graded left K-module $M = \mathsf{k}$ as provided by Lemma 3.41 (note that K is graded commutative). By assumption on K we know that $P_i = 0$ for $i \ll 0$ and that all P_i are finitely generated K-modules. Then $A \otimes_K^L \mathsf{k} \cong A \otimes_K \operatorname{tot}(P) = \operatorname{tot}(A \otimes_K P)$ in $D(\mathsf{k})$. Recall the decreasing filtration $F^l \operatorname{tot}(A \otimes_K P)$ of the dg k-module $\operatorname{tot}(A \otimes_K P)$. It is finite in our case. It gives rise to a spectral sequence $\{E_r^{ij}\}$ converging to $E_\infty^{ij} = \operatorname{gr}^i(H^{i+j}(\operatorname{tot}(A \otimes_K P)))$. Hence it is sufficient to prove that some page of the spectral sequence is finite dimensional.

Note that P_i is a graded free K-module, hence it is isomorphic to a direct sum of (finitely many) shifts of K as a dg K-module; hence $A \otimes_K P_i$ is isomorphic to a direct sum of shifts of A (as a dg A-module), and the E_1-page of our spectral sequence is given by

$$\begin{aligned} E_1^{ij} &= H^{i+j}(\operatorname{gr}^i \operatorname{tot}(A \otimes_K P)) = H^{i+j}([-i]A \otimes_K P_i) \\ &= H^j(A \otimes_K P_i) = H^j(A) \otimes_K P_i, \end{aligned}$$

with differential $d_1 : E_1^{ij} \to E_1^{i+1,j}$ equal to

$$\mathrm{id}_{H^j(A)} \otimes p_i : H^j(A) \otimes_K P_i \to H^j(A) \otimes_K P_{i+1}.$$

Let $H(A) \otimes_K P$ be the complex in $C(H(A))$ obtained from the complex P in $C(K)$ by extension of scalars along $K = H(K) \to H(A)$. The vertical differential d'' of its double complex $\operatorname{Dbl}(H(A) \otimes_K P)$ vanishes since $d_{H(A)} = 0$ and $d_{P_i} = 0$. If we forget it we have $E_1 = \operatorname{Dbl}(H(A) \otimes_K P)$. The vanishing of d'' then implies that $E_2 = H(E_1)$ has the

same dimension as $H(\mathrm{tot}(H(A)\otimes_K P))$. Note that $\mathrm{tot}(H(A)\otimes_K P) = H(A)\otimes_K \mathrm{tot}(P) \cong H(A)\otimes^L_K \mathsf{k}$. Lemma 3.41, now applied to $M = H(A)$ (which is finitely generated and of finite projective dimension as a K-module), shows that $\dim_{\mathsf{k}} H(H(A)\otimes^L_K \mathsf{k}) < \infty$. Hence E_2 is finite dimensional. □

4. Smoothness of equivariant derived categories

4.1. **Sheaves of dg modules over sheaves of dg algebras.** Let X be a topological space. We work with sheaves of modules over a fixed field k (up to and including Section 4.1.1 it could be a commutative ring; later on it will be $\mathbb{R}$) on X. Let $\mathcal{A} = \mathcal{A}_X$ be a sheaf of dg (k-)algebras on X. We denote by $\mathcal{M}od(\mathcal{A})$ the following dg (k-)category: Objects are dg (right) $\mathcal{A}$-modules (= sheaves of dg modules over the sheaf $\mathcal{A}$ of dg algebras), morphisms are $\mathcal{A}$-linear (and not necessarily compatible with the differentials), a morphism has degree n if it raises the degree by n, and the differentials on morphism spaces are defined in the usual way.

We define the abelian category $C(\mathcal{A})$, the (triangulated) homotopy category $\mathcal{H}(\mathcal{A}) := [\mathcal{M}od(\mathcal{A})]$ and the (triangulated) derived category $D(\mathcal{A})$ of dg $\mathcal{A}$-modules in the obvious way. Quasi-isomorphisms in $C(\mathcal{A})$ or $\mathcal{H}(\mathcal{A})$ are defined in the obvious way.

For example, the constant sheaf $\underline{\mathsf{k}} = \underline{\mathsf{k}}_X$ on X with stalk k is a sheaf of dg algebras and $D(\underline{\mathsf{k}}_X)$ is the usual (unbounded) derived category of sheaves of k-vector spaces on X. We denote this category sometimes by $D(X)$.

4.1.1. *Injective model structure on* $C(\mathcal{A})$. The following result is presumably known but we could not find a good reference (cf. [Bek00] and [Hov99b]).

Proposition 4.1. *There is a cofibrantly generated model structure on $C(\mathcal{A})$ such that the weak equivalences are the quasi-isomorphisms and the cofibrations are the monomorphisms.*

We call it the ***injective*** *model structure on $C(\mathcal{A})$.*

Proof. Our proof essentially coincides with the proof of [Bek00, Prop. 3.13] and is based on the result [Bek00, Thm. 1.7] by J. Smith. We therefore need to check the assumptions and conditions c0-c3 there, for the category $C(\mathcal{A})$, $\mathcal{W}$ the class of quasi-isomorphisms in $C(\mathcal{A})$, and I a suitable set of morphisms to be found.

If $j : U \hookrightarrow X$ is the inclusion of an open subset, define $S_{n,U} := [n]j_!j^*\mathcal{A}$ and $D_{n,U} := \mathrm{Cone}(\mathrm{id}_{S_{n,U}})$. Then, for $M \in C(\mathcal{A})$, we have canonical isomorphisms

$$(C(\mathcal{A}))(S_{n,U}, M) \xrightarrow{\sim} Z^{-n}(M(U)), \tag{4.1}$$

$$(C(\mathcal{A}))(D_{n,U}, M) \xrightarrow{\sim} M^{-n-1}(U). \tag{4.2}$$

The second equality implies that $C(\mathcal{A})$ is a Grothendieck category (cf. the proof of [KS06, Thm. 18.1.6]), and in particular locally presentable ([Bek00, Prop. 3.10]). In the same way $C(H(\mathcal{A}))$ is locally presentable.

Let $H : C(\mathcal{A}) \to C(H(\mathcal{A}))$ be the cohomology functor. It follows as in the proof of [Bek00, Prop. 3.13], using [Bek00, Prop. 1.15 and 1.18], that c3 is satisfied.

Let $\mathbf{Mono}$ be the class of monomorphisms in $C(\mathcal{A})$. Then [Bek00, Prop. 1.12] provides a set $I \subset \mathbf{Mono}$ such that $\mathbf{Mono} = I\text{-}\operatorname{cof}$. It is clear that $\mathbf{Mono} \cap \mathcal{W}$ is closed under pushouts and transfinite compositions (cf. proof of [Hov99b, Cor. 1.7]). This shows c2.

Condition c0 is obvious, so we are left to show c1, i.e. $I\text{-}\operatorname{inj} \subset \mathcal{W}$. Since $I\text{-}\operatorname{inj} = (I\text{-}\operatorname{cof})\text{-}\operatorname{inj} = \mathbf{Mono}\text{-}\operatorname{inj}$ we need to show that $\mathbf{Mono}\text{-}\operatorname{inj} \subset \mathcal{W}$. Let $(\varphi : M \to N) \in \mathbf{Mono}\text{-}\operatorname{inj}$. Let $x \in X$. We need to show that $\varphi_x : M_x \to N_x$ is a quasi-isomorphism. Any element of $Z^{-n}(N_x)$ comes from some $f \in Z^{-n}(N(U))$, for some open subset $U \subset X$ containing x. This element corresponds (use (4.1)) to the lower horizontal morphism in the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & M \arrow[d, "\varphi"] \\ S_{n,U} \arrow[r, "f"] \arrow[ur, dotted, "h"] & N \end{tikzcd}$$

which admits the indicated lift h. This shows that φ_x is surjective on cocycles.

Now assume that an element of $Z^{-n}(M_x)$ becomes a coboundary in N_x. Then there are an open subset $U \subset X$ containing x and elements $f \in Z^{-n}(M(U))$ and $g \in N^{-n-1}(U)$ such that f_x is the given cocycle and $\varphi(f) = d_N(g)$. If we interpret g and f as morphisms using (4.1) and (4.2) we obtain a commutative diagram

$$\begin{tikzcd} S_{n,U} \arrow[r, "f"] \arrow[d] & M \arrow[d, "\varphi"] \\ D_{n,U} \arrow[r, "g"] \arrow[ur, dotted, "h"] & N \end{tikzcd}$$

whose left vertical morphism is the obvious monomorphisms to the cone; then there is a lift h as indicated showing that the cocycle f is already a coboundary.

Now we can apply [Bek00, Thm. 1.7]. □

A dg $\mathcal{A}$-module I is called h-injective if all morphisms $N \to I$ in $C(\mathcal{A})$ with acyclic N are homotopic to zero, i.e. $(\mathcal{H}(\mathcal{A}))(N, I) = 0$. The arguments dual to those used in the proof of Lemma 2.6 show that fibrant objects in $C(\mathcal{A})$ are h-injective.

Denote by $\mathcal{H}(\mathcal{A})_{\text{h-inj}}$ (resp. $\mathcal{M}od(\mathcal{A})_{\text{h-inj}}$) the full subcategory of $\mathcal{H}(\mathcal{A})$ (resp. $\mathcal{M}od(\mathcal{A})$) consisting of h-injective dg $\mathcal{A}$-modules. Standard arguments show that the canonical functor

$$\mathcal{H}(\mathcal{A})_{\text{h-inj}} \xrightarrow{\sim} D(\mathcal{A}) \tag{4.3}$$

is a triangulated equivalence. Since $\mathcal{H}(\mathcal{A})_{\text{h-inj}} := [\mathcal{M}od(\mathcal{A})_{\text{h-inj}}]$ this means that $\mathcal{M}od(\mathcal{A})_{\text{h-inj}}$ is a dg enhancement of $D(\mathcal{A})$.

In Section 2.4 a (P) resolution of an object X was defined to be a trivial fibration $C \to X$ with C having property (P). All objects in the (projective) model categories considered there were fibrant. Now in the (injective) model category $C(\mathcal{A})$, all objects are cofibrant, so it is convenient to extend this definition and to say that a (P) resolution of an object X is a trivial cofibration $X \to F$ with F having property (P). Even if not mentioned we hope that it is always clear from the context which object is resolved.

We fix for any any dg $\mathcal{A}$-module N a fibrant (and hence h-injective) resolution, i. e. a monomorphic quasi-isomorphism $N \hookrightarrow \iota(N)$ in $C(\mathcal{A})$ with $\iota(N)$ fibrant (and hence h-injective). Then $N \mapsto \iota(N)$ extends to a functor

$$\iota : D(\mathcal{A}) \to \mathcal{H}(\mathcal{A})_{\text{h-inj}} \tag{4.4}$$

which is quasi-inverse to (4.3). We will use ι for (right-)deriving certain functors.

4.1.2. *Extension and restriction.* The structure morphism $\underline{\mathsf{k}} \to \mathcal{A}$ gives rise to dg functors $\operatorname{res} := \operatorname{res}^{\mathcal{A}}_{\underline{\mathsf{k}}}$ and $\operatorname{prod} := \operatorname{prod}^{\mathcal{A}}_{\underline{\mathsf{k}}}$,

$$\mathcal{M}od(\underline{\mathsf{k}}) \underset{\text{res}}{\overset{\text{prod}}{\rightleftarrows}} \mathcal{M}od(\mathcal{A}), \tag{4.5}$$

and there is an adjunction $(\operatorname{prod}, \operatorname{res})$ given by the obvious isomorphisms.

The following assumption on the structure morphism will be satisfied in our main applications.

(Str-Qiso) **The morphism $\underline{\mathsf{k}} \to \mathcal{A}$ is a quasi-isomorphism.**

Lemma 4.2. *Assume that* (Str-Qiso) *is satisfied. Then the adjunction* (4.5) *induces quasi-inverse equivalences*

$$D(\underline{\mathsf{k}}) \underset{\text{res}}{\overset{\text{prod}}{\underset{\sim}{\rightleftarrows}}} D(\mathcal{A})$$

of triangulated categories.

Proof. Let $M \in \mathcal{M}od(\underline{\mathsf{k}})$. Since we work over a field the dg functor $(M \otimes_{\underline{\mathsf{k}}} ?)$ preserves (acyclics and) quasi-isomorphisms (test on the stalks). If we apply this functor to the

quasi-isomorphism $\underline{\mathsf{k}} \to \mathcal{A}$ we see that the adjunction morphism $\varepsilon_M : M \to \operatorname{res}(M \otimes_{\underline{\mathsf{k}}} \mathcal{A})$ is a quasi-isomorphism.

Similarly, for $N \in \mathcal{M}od(\mathcal{A})$, the adjunction morphism $\delta_N : (\operatorname{res} N) \otimes_{\underline{\mathsf{k}}} \mathcal{A} \to N$ is a quasi-isomorphism: This is the case if and only if $\operatorname{res}(\delta_N)$ is a quasi-isomorphism; consider

$$\operatorname{res} N \xrightarrow{\varepsilon_{\operatorname{res} N}} \operatorname{res}((\operatorname{res} N) \otimes_{\underline{\mathsf{k}}} \mathcal{A}) \xrightarrow{\operatorname{res}(\delta_N)} \operatorname{res} N;$$

the composition is $\mathrm{id}_{\operatorname{res} N}$, and the first morphism is a quasi-isomorphism as observed above; hence $\operatorname{res}(\delta_N)$ is a quasi-isomorphism.

Note that prod (we work over a field) and res both preserve acyclics. Hence these two functors descend to triangulated functors between $D(\underline{\mathsf{k}})$ and $D(\mathcal{A})$ which are adjoint and quasi-inverse to each other by what we observed above. $\square$

Remark 4.3. *Assume that* (Str-Qiso) *is satisfied. One may ask whether* $(\operatorname{prod}, \operatorname{res})$ *defines a Quillen equivalence between* $C(\underline{\mathsf{k}})$ *and* $C(\mathcal{A})$ *if we equip each of these categories with the injective model structure from Proposition 4.1. One can use* [Fre09, Thm. 11.1.13] *to transfer the injective model structure on* $C(\underline{\mathsf{k}})$ *to a model structure on* $\mathcal{A}$ *such that we obtain a Quillen equivalence. For this model structure on* $C(\mathcal{A})$*, the weak equivalences are the quasi-isomorphisms, and the cofibrations are contained in the monomorphisms. We did not check whether we have equality there.*

4.1.3. *Standard functors for a decomposition into an open and a closed subspace.* We continue the above discussion by providing some preparations for the proof of Theorem 4.9 below.

Let $i : F \hookrightarrow X$ be a closed embedding and $j : U \hookrightarrow X$ the complementary open embedding. Let $\mathcal{A}_F := i^*\mathcal{A}$ and $\mathcal{A}_U := j^*\mathcal{A}$. Since the obvious dg functors i^*, $i_{\overset{*}{!}} := i_* = i_!$ and $j_!$, $j^{\overset{*}{!}} := j^! = j^*$ between $\mathcal{M}od(\mathcal{A}_F)$, $\mathcal{M}od(\mathcal{A}_X)$, and $\mathcal{M}od(\mathcal{A}_U)$ preserve acyclics they induce the horizontal functors in the following diagram:

$$\begin{tikzcd}
D(\underline{\mathsf{k}}_F) \arrow[r, "i_{\overset{*}{!}}"] & D(\underline{\mathsf{k}}_X) \arrow[l, bend right, "i^*"'] \arrow[r, "j^{\overset{*}{!}}"] & D(\underline{\mathsf{k}}_U) \arrow[l, bend right, "j_!"'] \\
D(\mathcal{A}_F) \arrow[u, "\operatorname{res}^{\mathcal{A}_F}_{\underline{\mathsf{k}}_F}"] \arrow[r, "i_{\overset{*}{!}}"] & D(\mathcal{A}_X) \arrow[u, "\operatorname{res}^{\mathcal{A}_X}_{\underline{\mathsf{k}}_X}"] \arrow[l, bend right, "i^*"'] \arrow[r, "j^{\overset{*}{!}}"] & D(\mathcal{A}_U) \arrow[u, "\operatorname{res}^{\mathcal{A}_U}_{\underline{\mathsf{k}}_U}"] \arrow[l, bend right, "j_!"']
\end{tikzcd}$$

The vertical functors are the obvious restriction functors (which preserve acyclics and hence descend trivially to the derived categories) along the respective structure morphisms as described in Section 4.1.2. In this diagram the four obvious squares commute, and we have adjunctions $(i^*, i_{\overset{*}{!}})$ and $(j_!, j^{\overset{*}{!}})$.

Since $i : F \to X$ is a closed embedding we have the dg functor $i^! : \mathcal{M}od(\mathcal{A}_X) \to \mathcal{M}od(\mathcal{A}_F)$. It preserves h-injectives since its left adjoint i_* preserves acyclics. We define $Ri^!$ to be the composition

$$Ri^! : D(\mathcal{A}_X) \xrightarrow{\iota} \mathcal{H}(\mathcal{A})_{\text{h-inj}} \xrightarrow{i^!} \mathcal{H}(\mathcal{A}_F)_{\text{h-inj}} \to D(\mathcal{A}_F).$$

Then we have an adjunction $(i_*, Ri^!)$. Similarly, we define $Rj_* : D(\mathcal{A}_U) \to D(\mathcal{A}_X)$ and obtain an adjunction (j^*, Rj_*), and we can do the same with $\underline{\mathsf{k}}$ instead of $\mathcal{A}$.

Now we assume that (Str-Qiso) is satisfied. Then also $\underline{\mathsf{k}}_F \to \mathcal{A}_F$ and $\underline{\mathsf{k}}_U \to \mathcal{A}_U$ are quasi-isomorphisms. We obtain the diagram

(4.6)
$$\begin{tikzcd}
D(\underline{\mathsf{k}}_F) \arrow[r, "i_*"] & D(\underline{\mathsf{k}}_X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^*"] & D(\underline{\mathsf{k}}_U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \\
D(\mathcal{A}_F) \arrow[u, "\mathrm{res}^{\mathcal{A}_F}_{\underline{\mathsf{k}}_F}", "\sim"'] \arrow[r, "i_*"] & D(\mathcal{A}_X) \arrow[u, "\mathrm{res}^{\mathcal{A}_X}_{\underline{\mathsf{k}}_X}", "\sim"'] \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^*"] & D(\mathcal{A}_U). \arrow[u, "\mathrm{res}^{\mathcal{A}_U}_{\underline{\mathsf{k}}_U}", "\sim"'] \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}$$

The vertical arrows are equivalences (Lemma 4.2), and the square with horizontal sides $Ri^!$ (resp. Rj_*) commutes (up to natural isomorphisms) since res preserves h-injectives (its left adjoint prod preserves acyclics).

Lemma 4.4. *Assume that* (Str-Qiso) *is satisfied.*

(a) Let $\mathcal{F} \in C(\underline{\mathsf{k}}_F)$ be an object. Then there is a fibrant object $\mathcal{F}' \in C(\mathcal{A}_F)$ and a monomorphic quasi-isomorphism $\mathcal{F} \to \mathrm{res}(\mathcal{F}')$ in $C(\underline{\mathsf{k}}_F)$. This morphism remains a monomorphic quasi-isomorphism under i_*. Moreover, $i_*\mathcal{F}'$ is h-injective (and fibrant) and $\mathrm{res}(i_*\mathcal{F}') \cong i_* \mathrm{res}(\mathcal{F}')$ in $C(\underline{\mathsf{k}}_F)$.

(b) Let $\mathcal{U} \in C(\underline{\mathsf{k}}_U)$ be given. Then there is $\mathcal{V} \in C(\mathcal{A}_X)$ fibrant together with a monomorphic quasi-isomorphism $j_!(\mathcal{U}) \hookrightarrow \mathrm{res}(\mathcal{V})$ and a fibrant resolution $\mathcal{U} \otimes_{\underline{\mathsf{k}}_U} \mathcal{A}_U \hookrightarrow j^*(\mathcal{V})$ in $C(\mathcal{A}_U)$ Moreover, $\mathcal{U} \xrightarrow{\sim} j^* j_! \mathcal{U} \hookrightarrow j^* \mathrm{res}(\mathcal{V}) \cong \mathrm{res}\, j^*(\mathcal{V})$ is a monomorphic quasi-isomorphism in $C(\underline{\mathsf{k}}_U)$.

Proof. (a): The object $\mathrm{prod}(\mathcal{F}) = \mathcal{F} \otimes_{\underline{\mathsf{k}}_F} \mathcal{A}_F$ has a fibrant resolution $\mathcal{F} \otimes_{\underline{\mathsf{k}}_F} \mathcal{A}_F \to \mathcal{F}'$. Apply res (which preserves monomorphism and quasi-isomorphisms) and use the monomorphic quasi-isomorphism $\varepsilon_{\mathcal{F}} : \mathcal{F} \to \mathrm{res}(\mathcal{F} \otimes_{\underline{\mathsf{k}}_F} \mathcal{A}_F)$ (test on stalks: on the stalk at $x \in X$ this morphism is given by applying the exact functor $(\mathcal{F}_x \otimes_{\underline{\mathsf{k}}_{F,x}} ?)$ to $\mathsf{k} = \underline{\mathsf{k}}_{F,x} \to \mathcal{A}_x$; the latter morphism is a quasi-isomorphisms and injective since $\mathsf{k} = Z^0(\mathsf{k}) = H^0(\mathsf{k})$).

Obviously, i_* preserves monomorphisms and quasi-isomorphisms. Note that i_* preserves h-injectives, since its left adjoint i^* preserves acyclics (since i^* preserves monomorphic

quasi-isomorphisms (= trivial cofibrations), $i_{\ast}$ preserves fibrations). It is clear that res and $i_{\ast}$ commute.

(b) The first statement is proved as above, using a fibrant resolution $j_!(\mathcal{U}) \otimes_{\underline{\mathsf{k}}_X} \mathcal{A}_X \hookrightarrow \mathcal{V}$. The left adjoint $j_!$ of $j^{\ast}$ preserves trivial cofibrations, hence $j^{\ast}$ preserves fibrations. In particular $j^{\ast}(\mathcal{V})$ is fibrant. If we apply $j^{\ast}$ to the cofibrant resolution $j_!(\mathcal{U}) \otimes_{\underline{\mathsf{k}}_X} \mathcal{A}_X \hookrightarrow \mathcal{V}$ we obtain a monomorphic quasi-isomorphism $\mathcal{U} \otimes_{\underline{\mathsf{k}}_U} \mathcal{A}_U \hookrightarrow j^{\ast}(\mathcal{V})$. The last statement is clear since again $j^{\ast}$ preserves trivial cofibrations (= monomorphic quasi-isomorphisms). □

4.2. **Refined enhancements of equivariant derived categories.** Let G be a connected complex affine algebraic group and X a complex G-variety. We usually equip G and X with the classical topology. In the following, our field k will be $\mathbb{R}$.

Let $p : EG \to BG = EG/G$ be a universal G-principal fiber bundle such that BG is an ∞-dimensional manifold in the sense of [BL94, 12.2] or a smooth (paracompact) manifold of finite dimension, and such that EG is ∞-acyclic. Such a bundle exists by [BL94, 12.4.2.a], and we can and will assume that EG is open-locally pre-∞-acyclic (as defined in the Appendix A) and that BG is locally contractible. There is a sheaf Ω_{BG} of graded commutative dg ($\mathbb{R}$-)algebras (the de Rham sheaf) on BG such that the obvious morphism $\underline{\mathbb{R}}_{BG} \to \Omega_{BG}$ is a quasi-isomorphism and each Ω^i_{BG} is soft and acyclic ([BL94, 12.2.3]).

Let $X_G := EG \times_G X$ be the quotient of $EG \times X$ by the diagonal G action. Since $EG \times X \to X$ is an ∞-acyclic resolution, the bounded constructible G-equivariant derived category $D^b_{G,c}(X)$ can be viewed as a full subcategory of $D(X_G)$ [BL94, 2.9.9]: it consists of those object whose pullback to $EG \times X$ is isomorphic to the pullback of an object of $D^b_c(X)$. Let

$$c : X_G \to BG = \mathrm{pt}_G$$

be the obvious morphism. Since the obvious morphism $\underline{\mathbb{R}}_{X_G} \to c^*(\Omega_{BG})$ is a quasi-isomorphism, Lemma 4.2 shows that $D(X_G)$ and $D(c^*(\Omega_{BG}))$ are equivalent as triangulated categories. Hence we can view $D^b_{G,c}(X)$ as a full subcategory of $D(c^*(\Omega_{BG}))$. Recall that $\mathcal{M}od(c^*(\Omega_{BG}))_{\text{h-inj}}$ is a dg enhancement of $D(c^*(\Omega_{BG}))$. Let $\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ be the full subcategory of $\mathcal{M}od(c^*(\Omega_{BG}))_{\text{h-inj}}$ consisting of objects that are isomorphic (in $D(c^*(\Omega_{BG}))$) to an object of $D^b_{G,c}(X)$. Then $\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ is a dg enhancement of $D^b_{G,c}(X)$. Note that $\mathcal{M}od(c^*(\Omega_{BG}))$ is a dg $\Gamma(\Omega_{BG})$-category in the obvious way (and that $\Gamma(\Omega_{BG})$ is graded commutative), and therefore $\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ is a dg $\Gamma(\Omega_{BG})$-category.

Definition 4.5. *We say that* X *is* ***(homologically)*** G ***-smooth*** *(or more precisely* $D^b_{G,c}$ ***-smooth****), if* $\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ *is a smooth*[2] *dg* $\Gamma(\Omega_{BG})$ *-category.*

Remark 4.6. *We could not find a good way of proving that this definition is independent of the choice of* $p : EG \to BG$*, so we should more precisely speak of* G *-smoothness with respect to* p*. Instead of doing this we fix* p *as above for the rest of the article. This is justified by the fact that in the case of interest to us (where* G *has only finitely many orbits in* X *and all stabilizers are connected), independence of* p *will be a consequence of Theorems 4.9 and 4.8 (and Propositions 4.10 and 4.11).*

The following Lemma explains that G-smoothness can be tested on the dg endomorphisms of a classical generator.

Lemma 4.7. *Assume that there is an object* $E \in \mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ *such that* E *is a classical generator of* $[\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}] \xrightarrow{\sim} D^b_{G,c}(X)$*. Then* X *is* G *-smooth if and only if* $(\mathcal{M}od(c^*(\Omega_{BG})))(E)$ *is* $\Gamma(\Omega_{BG})$ *-smooth.*

Proof. Note that $D^b_{G,c}(X)$ is Karoubian since it has a bounded t-structure [LC07]. Then the result follows from Corollary 3.19. □

4.3. **Smoothness of homogeneous spaces.** Let G be a connected complex affine algebraic group and let $H \subset G$ be a closed subgroup. We discuss the smoothness of the G-variety $X = G/H$. Let $p : EG \to BG$ be as above. Then $EG \to EG/H$ is a universal H-principal fiber bundle, and we define $BH := EG/H$. Then

$$X_G = EG \times_G G/H = (EG \times_G G)/H = EG/H = BH$$

canonically, so $c : X_G = BH \to BG$. Since c is a locally trivial bundle with fiber G/H we can equip BH with the structure of a (possibly ∞-dimensional) manifold. In particular this defines the de Rham sheaf Ω_{BH} on BH. Pullback of differential forms [BL94, 12.2.6] yields a morphism

$$\Gamma(\Omega_{BG}) \to \Gamma(\Omega_{BH})$$

of (graded commutative) dg $\mathbb{R}$-algebras. Taking cohomology we get the usual morphism

$$H_G(\text{pt}) \to H_G(G/H) = H_H(\text{pt})$$

on equivariant cohomology (since BG and BH are locally contractible and paracompact, sheaf cohomology coincides with singular cohomology).

[2] Here we implicitly replace $\mathcal{M}od_G(c^*(\Omega_{BG}))_{\text{h-inj}}$ by a dg $\Gamma(\Omega_{BG})$-equivalent dg $\Gamma(\Omega_{BG})$-subcategory which is small.

Theorem 4.8. *Keep the above assumptions and assume in addition that H is connected. The following conditions are equivalent:*

(a) $X = G/H$ *is* G*-smooth.*
(b) $\Gamma(\Omega_{X_G}) = \Gamma(\Omega_{BH})$ *is* $\Gamma(\Omega_{BG})$*-smooth.*
(c) $\Gamma(\Omega_{BG}) \to \Gamma(\Omega_{BH})$ *is a quasi-isomorphism.*
(d) $H_G(pt) \to H_G(G/H) = H_H(pt)$ *is an isomorphism.*
(e) If E *is an h-injective dg* $c^*(\Omega_{BG})$*-module that is isomorphic to* Ω_{X_G} *in* $D(c^*(\Omega_{BG}))$*, the structure morphism* $\Gamma(\Omega_{BG}) \to (\mathcal{M}od(c^*(\Omega_{BG})))(E)$ *is a quasi-isomorphism.*

(More equivalent conditions can be found in Propositions 4.10 and 4.11.)

Proof. The equivalence of the two conditions (c) and (d) is obvious.

Define the dg $\Gamma(\Omega_{BG})$ category $\mathcal{M} := \mathcal{M}od(c^*(\Omega_{BG}))$. Since H is connected the constant sheaf $\underline{\mathbb{R}}_{X_G}$ on X_G is a classical generator of $D^b_{G,c}(X)$ considered as a subcategory of $D(X_G)$ (use [BL94, Induction equivalence 2.6.3 and Prop. 2.7.2]). Hence $c^*(\Omega_{BG})$) is a classical generator of $D^b_{G,c}(X)$ considered as a subcategory of $D(c^*(\Omega_{BG}))$. Let E be an h-injective dg $c^*(\Omega_{BG})$-module that is isomorphic to Ω_{X_G} in $D(c^*(\Omega_{BG}))$. Then there is a quasi-isomorphism $\varepsilon : \Omega_{X_G} \to E$ in $C(c^*(\Omega_{BG}))$. Then $X = G/H$ is G-smooth if and only if $\mathcal{M}(E)$ is $\Gamma(\Omega_{BG})$-smooth, by Lemma 4.7.

Pullback of differential forms along $c : X_G \to BG$ defines a (monomorphic) quasi-isomorphism of sheaves of dg algebras

$$c^\sharp : c^*(\Omega_{BG}) \to \Omega_{X_G}$$

since both sheaves are resolutions of $\underline{\mathbb{R}}_{X_G}$.

Consider the dg $\mathcal{M}(\Omega_{X_G}) \otimes_{\Gamma(\Omega_{BG})} \mathcal{M}(E)^{\mathrm{op}}$-module $\mathcal{M}(\Omega_{X_G}, E)$,

$$\mathcal{M}(E) \curvearrowright \mathcal{M}(\Omega_{X_G}, E) \curvearrowleft \mathcal{M}(\Omega_{X_G}).$$

Restriction along the morphism of dg $\Gamma(\Omega_{BG})$-algebras

$$\lambda : \Gamma(\Omega_{X_G}) \to \mathcal{M}(\Omega_{X_G}), \quad \omega \mapsto (\omega \wedge ?),$$

turns $\mathcal{M}(\Omega_{X_G}, E)$ into an dg $\Gamma(\Omega_{X_G}) \otimes_{\Gamma(\Omega_{BG})} \mathcal{M}(E)^{\mathrm{op}}$-module,

$$\mathcal{M}(E) \curvearrowright \mathcal{M}(\Omega_{X_G}, E) \curvearrowleft \Gamma(\Omega_{X_G}).$$

Note that $\varepsilon \in Z^0(\mathcal{M}(\Omega_{X_G}, E))$. We claim that the action maps

$$(? \circ \varepsilon) : \mathcal{M}(E) \to \mathcal{M}(\Omega_{X_G}, E) \quad \text{and} \tag{4.7}$$

$$(\varepsilon \circ ?) \circ \lambda : \Gamma(\Omega_{X_G}) \to \mathcal{M}(\Omega_{X_G}, E) \tag{4.8}$$

are quasi-isomorphisms of dg $\Gamma(\Omega_{BG})$-modules. (The idea of the proof of this claim is taken from [Soe10].) This is obvious for (4.7) since E is h-injective and ε is an isomorphism in

the derived category. The action map (4.8) appears as the upper horizontal composition in the following diagram in $C(\Gamma(\Omega_{BG}))$:

$$\begin{tikzcd}
\Gamma(\Omega_{X_G}) \arrow[r, "\lambda"] \arrow[d, equal] & \mathcal{M}(\Omega_{X_G}, \Omega_{X_G}) \arrow[r, "\varepsilon\circ ?"] & \mathcal{M}(\Omega_{X_G}, E) \arrow[dd, "?\circ c^\sharp"] \\
\Gamma(\Omega_{X_G}) \arrow[r, "\Gamma(\varepsilon)"] \arrow[d, "\sim"', "\mathrm{can}"] & \Gamma(E) \arrow[d, "\sim"', "\mathrm{can}"] & \\
\mathcal{M}(c^*(\Omega_{BG}), \Omega_{X_G}) \arrow[r, "\varepsilon\circ ?"] & \mathcal{M}(c^*(\Omega_{BG}), E) \arrow[r, equal] & \mathcal{M}(c^*(\Omega_{BG}), E)
\end{tikzcd}$$

Note first that this diagram is commutative: For $\omega \in \Gamma(\Omega_{X_G})$ we have

$$(\varepsilon \circ \lambda(\omega) \circ c^\sharp)(1) = \varepsilon(\omega) = ((\varepsilon \circ \mathrm{can}(\omega))(1).$$

Since E is h-injective and $c^\sharp$ is a quasi-isomorphism, the right vertical morphism $(? \circ c^\sharp)$ is a quasi-isomorphism. So we have to show that $\Gamma(\varepsilon)$ is a quasi-isomorphism. It is enough to show that $\Gamma(\mathrm{res}(\varepsilon)) : \Gamma(\mathrm{res}(\Omega_{X_G})) \to \Gamma(\mathrm{res}(E))$ is a quasi-isomorphism, where $\mathrm{res} = \mathrm{res}^{c^*(\Omega_{BG})}_{\underline{\mathbb{R}}_{X_G}}$.

The obvious map $\underline{\mathbb{R}}_{X_G} \to \mathrm{res}(\Omega_{X_G})$ and its composition $\underline{\mathbb{R}}_{X_G} \to \mathrm{res}(E)$ with $\mathrm{res}(\varepsilon)$ are quasi-isomorphisms. Since $\Omega_{X_G} = \mathrm{res}(\Omega_{X_G})$ is soft and X_G is paracompact, $\Gamma(\mathrm{res}(\Omega_{X_G}))$ computes the sheaf cohomology $H(X_G; \underline{\mathbb{R}}_{X_G})$, and so does $\Gamma(\mathrm{res}(E))$ (note that $\mathrm{res}(E)$ is h-injective since the left adjoint of res preserves acyclics). This shows that $\Gamma(\varepsilon)$ is a quasi-isomorphism and proves that (4.8) is a quasi-isomorphism.

The fact that (4.7) and (4.8) are quasi-isomorphisms has the following two consequences: Firstly, the conditions (c) and (e) are equivalent. Secondly, $\Gamma(\Omega_{X_G})$ is $\Gamma(\Omega_{BG})$-smooth if and only if $\mathcal{M}(E)$ is $\Gamma(\Omega_{BG})$-smooth: this follows from [Lun10, Lemma 2.14] and Lemma 3.12 (alternatively, it is easy to see that $\mathcal{M}(E)$ and $\Gamma(\Omega_{X_G})$ are dg Morita equivalent, and then one can use Theorem 3.17). Hence (a) and (b) are equivalent.

It remains to show that (b) and (c) are equivalent. We know that $K := H(\Gamma(\Omega_{BG})) = H_G(\mathrm{pt})$ is a polynomial ring over $\mathbb{R}$ in finitely many variables of positive even degrees (cf. proof of Proposition 4.10 below). Since $\Gamma(\Omega_{BG})$ is graded commutative there is a quasi-isomorphism of dg ($\mathbb{R}$-)algebras $K \to \Gamma(\Omega_{BG})$ inducing the identity on cohomology. By Theorem 3.30, part (BC3), $\Gamma(\Omega_{BG})$-smoothness of $\Gamma(\Omega_{X_G}) = \Gamma(\Omega_{BH})$ is equivalent to K-smoothness of $\Gamma(\Omega_{BH})$. This latter condition is equivalent to $K \to \Gamma(\Omega_{BH})$ being a quasi-isomorphism, by Proposition 3.43 (the assumptions there are satisfied by the proof of Proposition 4.10), hence to $\Gamma(\Omega_{BG}) \to \Gamma(\Omega_{BH})$ being a quasi-isomorphism. □

4.4. **Reduction to homogeneous spaces.**

Theorem 4.9. *Let G be a connected complex affine algebraic group and X a complex G-variety. Assume that X consists of finitely many G-orbits and that all stabilizers (in G of points in X) are connected. The following conditions are equivalent:*

(a) X is G-smooth.

(b) All G-orbits in X are G-smooth.

Proof. Assume that U is an open G-orbit in X, and let F be its complement.

Let $\mathcal{A}_{X_G} = c^*(\Omega_{BG})$, where $c : X_G \to BG$. Diagram (4.6) for $\mathsf{k} = \mathbb{R}$ yields the following diagram (we write $i : F_G \to X_G$ instead of i_G etc.):

$$\begin{tikzcd}
D(F_G) \arrow[r, "i_{\overset{*}{!}}"] & D(X_G) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^{\overset{*}{!}}"] & D(U_G) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"] \\
D(\mathcal{A}_{F_G}) \arrow[u, "\operatorname{res}^{\mathcal{A}_{F_G}}_{\underline{\mathbb{R}}_{F_G}}", "\sim"'] \arrow[r, "i_{\overset{*}{!}}"] & D(\mathcal{A}_{X_G}) \arrow[u, "\operatorname{res}^{\mathcal{A}_{X_G}}_{\underline{\mathbb{R}}_{X_G}}", "\sim"'] \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^{\overset{*}{!}}"] & D(\mathcal{A}_{U_G}) \arrow[u, "\operatorname{res}^{\mathcal{A}_{U_G}}_{\underline{\mathbb{R}}_{U_G}}", "\sim"'] \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}$$

We have explained its properties above. We claim that all functors in the upper row induce the following functors between the equivariant derived categories (and then it is clear that they coincide with the usual functors defined on the equivariant level).

$$\begin{tikzcd}
D^b_{G,c}(F) \arrow[r, "i_{\overset{*}{!}}"] & D^b_{G,c}(X) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "Ri^!"] \arrow[r, "j^{\overset{*}{!}}"] & D^b_{G,c}(U) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "Rj_*"]
\end{tikzcd}$$

This is obvious for i^* and $j^{\overset{*}{!}}$. Theorem A.3 implies that it is true for Rj_* and $i_{\overset{*}{!}}$ (for constructibility use [Kal05], which implies that the decomposition of X into G-orbits is a Whitney stratification). To get the result for $Ri^!$ let $\mathcal{F} \in D^b_{G,c}(X) \subset D(X_G)$, and consider the triangle

$$i_{\overset{*}{!}}(Ri^!(\mathcal{F})) \to \mathcal{F} \to Rj_*(j^{\overset{*}{!}}(\mathcal{F})) \to [1]i_{\overset{*}{!}}(Ri^!(\mathcal{F})).$$

We already know that $\mathcal{F}$ and $Rj_*(j^{\overset{*}{!}}(\mathcal{F}))$ are in $D^b_{G,c}(X)$. Hence $i_{\overset{*}{!}}(Ri^!(\mathcal{F})) \in D^b_{G,c}(X)$ and $Ri^!(\mathcal{F}) \xleftarrow{\sim} i^*(i_{\overset{*}{!}}(Ri^!(\mathcal{F}))) \in D^b_{G,c}(F)$. Similarly one gets the result for $j_!$.

Let $\mathcal{U} := \underline{\mathbb{R}}_{U_G}$. We have seen at the beginning of the proof of Theorem 4.8 that this is a classical generator of $D^b_{G,c}(U)$. If $\mathcal{F}$ is a classical generator of $D^b_{G,c}(F)$, then $\{i_{\overset{*}{!}}\mathcal{F}, j_!\mathcal{U}\}$ (or $i_{\overset{*}{!}}\mathcal{F} \oplus j_!\mathcal{U}$) classically generates $D^b_{G,c}(X)$: Any object $\mathcal{X} \in D^b_{G,c}(X)$ fits into a (distinguished) triangle

$$j_!j^{\overset{*}{!}}\mathcal{X} \to \mathcal{X} \to i_{\overset{*}{!}}i^*\mathcal{X} \to [1]j_!j^{\overset{*}{!}}\mathcal{X},$$

and $j^*\mathcal{X} \in D^b_{G,c}(U)$ (resp. $i^*\mathcal{X} \in D^b_{G,c}(F)$) is in the subcategory classically generated by $\mathcal{U}$ (resp. $\mathcal{F}$). This argument and an induction on the number of G-orbits also shows that $D^b_{G,c}(F)$ has a classical generator; we fix such a generator $\mathcal{F}$.

From Lemma 4.4 we obtain (where res denotes the obvious restriction functors): There is an h-injective dg $\mathcal{A}_{F_G}$-module $\mathcal{F}'$ such that $\operatorname{res}(\mathcal{F}') \cong \mathcal{F}$ in $D(F_G)$ and such that $i_*\mathcal{F}'$ is h-injective and $\operatorname{res}(i_*\mathcal{F}') \cong i_*\mathcal{F}$ in $D(X_G)$. There is an h-injective dg $\mathcal{A}_{X_G}$-module $\mathcal{V}$ such that $\operatorname{res}(\mathcal{V}) \cong j_!\mathcal{U}$ in $D(X_G)$ and such that $j^*(\mathcal{V})$ is h-injective and satisfies $\mathcal{A}_{U_G} = \mathcal{U} \otimes_{\underline{\mathbb{R}}_{U_G}} \mathcal{A}_{U_G} \cong j^*(\mathcal{V})$ in $D(\mathcal{A}_{U_G})$ and $\operatorname{res}(j^*(\mathcal{V})) \cong \mathcal{U}$ in $D(U_G)$.

Let $\mathcal{E}$ be the full subcategory of $\mathcal{M}od(\mathcal{A}_{X_G})$ whose objects are $i_*\mathcal{F}'$ and $\mathcal{V}$. We write this $\Gamma(\Omega_{BG})$-category $\mathcal{E}$ symbolically as

$$\begin{bmatrix} \mathcal{E}(i_*\mathcal{F}') & \mathcal{E}(\mathcal{V}, i_*\mathcal{F}') \\ \mathcal{E}(i_*\mathcal{F}', \mathcal{V}) & \mathcal{E}(\mathcal{V}) \end{bmatrix}.$$

From Remark 3.25 and Lemma 4.7 (applied to $i_*\mathcal{F}' \oplus \mathcal{V}$) we see that X is G-smooth if and only if $\mathcal{E}$ is $\Gamma(\Omega_{BG})$-smooth.

Define $\mathcal{E}' \subset \mathcal{E}$ to be the subcategory with the same objects and morphisms except that $\mathcal{E}'(\mathcal{V}, i_*\mathcal{F}') := 0$, symbolically

$$\mathcal{E}' = \begin{bmatrix} \mathcal{E}(i_*\mathcal{F}') & 0 \\ \mathcal{E}(i_*\mathcal{F}', \mathcal{V}) & \mathcal{E}(\mathcal{V}) \end{bmatrix}.$$

Then the obvious morphism $\mathcal{E} \to \mathcal{E}'$ of dg $\Gamma(\Omega_{BG})$-categories is a quasi-equivalence: We only need to show that $\mathcal{E}(\mathcal{V}, i_*\mathcal{F}')$ is acyclic. But

$$\begin{aligned} H(\mathcal{E}(\mathcal{V}, i_*\mathcal{F}')) &= (\mathcal{H}(\mathcal{A}_{X_G}))(\mathcal{V}, i_*\mathcal{F}') \\ (\text{since } i_*\mathcal{F}' \text{ is h-injective}) &\xrightarrow{\sim} (D(\mathcal{A}_{X_G}))(\mathcal{V}, i_*\mathcal{F}') \\ (\text{since res is an equivalence}) &\xrightarrow{\sim} (D(X_G))(\operatorname{res}\mathcal{V}, \operatorname{res} i_*\mathcal{F}') \\ &\cong (D(X_G))(j_!\mathcal{U}, i_*\mathcal{F}) \\ &\cong (D(X_G))(i^* j_!\mathcal{U}, \mathcal{F}) \\ (\text{since } i^* j_! = 0) &= 0. \end{aligned}$$

Lemma 3.12 and Theorem 3.24 show that G-smoothness of X is equivalent to

(a') $\mathcal{E}'(i_*\mathcal{F}')$ is $\Gamma(\Omega_{BG})$-smooth, and

(b') $\mathcal{E}'(\mathcal{V})$ is $\Gamma(\Omega_{BG})$-smooth, and

(c') $\mathcal{E}'(i_*\mathcal{F}', \mathcal{V})$ is $\Gamma(\Omega_{BG})$-good as a dg $\mathcal{E}'(i_*\mathcal{F}') \otimes_{\Gamma(\Omega_{BG})} \mathcal{E}'(\mathcal{V})^{\mathrm{op}}$-module.

We claim that these three conditions are equivalent to the following two conditions

(a") F is G-smooth, and

(b") U is G-smooth.

(a') $\Leftrightarrow$ (a"): Condition (a') is equivalent to $(\mathcal{M}od(\mathcal{A}_{X_F}))(\mathcal{F}')$ being $\Gamma(\Omega_{BG})$-smooth (since i_* is fully faithful), and hence to G-smoothness of F, by Lemma 4.7.

(b') $\Leftrightarrow$ (b"): Note that

$$j^* : \mathcal{E}'(\mathcal{V}) = (\mathcal{M}od(\mathcal{A}_{X_G}))(\mathcal{V}) \to (\mathcal{M}od(\mathcal{A}_{U_G}))(j^*\mathcal{V}) \tag{4.9}$$

is a quasi-isomorphism: on the p-th cohomology it is given by

$$j^* : (D(\mathcal{A}_{X_G}))(\mathcal{V}, [p]\mathcal{V}) \to (D(\mathcal{A}_{U_G}))(j^*\mathcal{V}, [p]j^*\mathcal{V})$$

which becomes identified (using the equivalences res, $\operatorname{res} j^* \cong j^* \operatorname{res}$, and $\operatorname{res}\mathcal{V} \cong j_!\mathcal{U}$) with

$$j^* : (D(X_G))(j_!\mathcal{U}, [p]j_!\mathcal{U}) \to (D(U_G))(j^*j_!\mathcal{U}, [p]j^*j_!\mathcal{U})$$

and is an isomorphism since $j_!$ is fully faithful. Hence condition (b') is equivalent to $\Gamma(\Omega_{BG})$-smoothness of $(\mathcal{M}od(\mathcal{A}_{U_G}))(j^*\mathcal{V})$ (Lemma 3.12). Now again use Lemma 4.7 and the fact that $j^*\mathcal{V}$ is h-injective.

(b") $\Rightarrow$ (c'): Assume that (b") holds. Since $\mathcal{A}_{U_G} \cong j^*(\mathcal{V})$ in $D(\mathcal{A}_{U_G})$, Theorem 4.8 implies that the structure morphism $\Gamma(\Omega_{BG}) \to (\mathcal{M}od(\mathcal{A}_{U_G}))(j^*(\mathcal{V}))$ is a quasi-isomorphism. By the above quasi-isomorphism (4.9) this is equivalent to $\Gamma(\Omega_{BG}) \to \mathcal{E}'(\mathcal{V})$ being a quasi-isomorphism. Hence Corollary 3.5 shows that condition (c') is equivalent to

$$\mathcal{E}'(i_*\mathcal{F}', \mathcal{V}) \in \operatorname{per}(\mathcal{E}'(i_*\mathcal{F}')).$$

Applying the adjunction $(i_*, i^!)$ (on the dg level) and using that i_* is fully faithful we see that this is equivalent to

$$(\mathcal{M}od(\mathcal{A}_{X_F}))(\mathcal{F}', i^!\mathcal{V}) \in \operatorname{per}((\mathcal{M}od(\mathcal{A}_{F_G}))(\mathcal{F}')). \tag{4.10}$$

If we view $D^b_{G,c}(F)$ as a full subcategory of $D(\mathcal{A}_{F_G})$, then equivalence (4.4) (cf. Lemma 4.7) and Corollary 3.19 show that

$$D^b_{G,c}(F) \xrightarrow{\sim} \operatorname{per}((\mathcal{M}od(\mathcal{A}_{F_G}))(\mathcal{F}')),$$
$$\mathcal{G} \mapsto (\mathcal{M}od(\mathcal{A}))(\mathcal{F}', \iota(\mathcal{G})),$$

is an equivalence.

Since $Ri^!$ preserves the equivariant derived categories and $\mathcal{V}$ is h-injective, we have $i^!(\mathcal{V}) \cong Ri^!(\mathcal{V}) \in D^b_{G,c}(F)$. Since i_* is left adjoint to $i^!$ and preserves acyclics, $i^!(\mathcal{V})$ is h-injective. Hence $i^!(\mathcal{V}) \cong \iota(i^!(\mathcal{V}))$ already in the homotopy category and therefore

$$(\mathcal{M}od(\mathcal{A}))(\mathcal{F}', i^!(\mathcal{V})) \cong (\mathcal{M}od(\mathcal{A}))(\mathcal{F}', \iota(i^!(\mathcal{V}))) \in \operatorname{per}((\mathcal{M}od(\mathcal{A}_{F_G}))(\mathcal{F}')).$$

This shows (4.10) and hence that condition (c') is satisfied.

These arguments show that X is G-smooth if and only if U and F are G-smooth. An induction on the number of G-orbits in X finishes the proof. $\square$

4.5. **Results concerning the case of a homogeneous space.** Let H be a closed connected subgroup of a connected (real) Lie group G. The inclusion morphism $H \to G$ gives rise to the morphism $H_G(\mathrm{pt}) \to H_H(\mathrm{pt})$ on equivariant cohomology which is a morphism of (graded commutative) dg $\mathbb{R}$-algebras (with differential zero).

Proposition 4.10. *Let H be a closed connected subgroup of a connected Lie group G. Then the following conditions are equivalent:*

(a) *$H_H(pt)$ is a smooth dg $H_G(pt)$-algebra;*
(b) *$H_G(pt) \to H_H(pt)$ is an isomorphism;*
(c) *a/any maximal compact subgroup of H is a maximal compact subgroup of G.*
(d) *$H(G/H) := H^*(G/H; \mathbb{R}) = \mathbb{R}$.*

Proof. Recall the following facts (see [Hoc65, Bor95]): Any connected (real) Lie group G' has maximal compact subgroups and any compact subgroup is contained in one of them; they are connected and any two of them are conjugate by an inner automorphism. If K' is a maximal compact subgroup, G' is homeomorphic to $K' \times \mathbb{R}^l$ for some l, and the quotient G'/K' is homeomorphic to $\mathbb{R}^l$.

We write H_G instead of $H_G(\mathrm{pt})$, and similar for other groups. Let M be a maximal compact subgroup of H, and L a maximal compact subgroup of G containing M. Let T be a maximal torus in M, and S a maximal torus in L containing T:

$$\begin{array}{ccccc} S & \subset & L & \subset & G \\ \cup & & \cup & & \cup \\ T & \subset & M & \subset & H \end{array}$$

Since $G \cong L \times \mathbb{R}^g$ and $G/L \cong \mathbb{R}^g$ for some $g \in \mathbb{N}$ we have $H_G = H_L$. Let W_L be the Weyl group of (L, S). It acts on $\mathrm{Sym}((\mathrm{Lie}\, S)^*)$, the space of real valued polynomial functions on the Lie algebra $\mathrm{Lie}\, S$ of S. We have canonically $\mathrm{Sym}((\mathrm{Lie}\, S)^*) = H_S$. From [Bor53, §27, §28] we know that $H_L = H_S^{W_L}$ and that this is a polynomial ring (over $\mathbb{R}$) in $s := \dim_{\mathbb{R}} S$ variables of positive even degrees; similarly $H_H = H_M = H_T^{W_M}$ is a polynomial ring in $t := \dim_{\mathbb{R}} T$ variables of positive even degrees; furthermore, the inclusion $\mathrm{Lie}\, T \subset \mathrm{Lie}\, S$ induces a morphism $H_S^{W_L} \to H_T^{W_M}$ which coincides with $H_G \to H_H$ under our identifications.

Now it is clear that (c) implies (b). Conversely, (b) implies that $S = T$ (if $T \subsetneq S$, choose $0 \neq \chi \in (\mathrm{Lie}\, S)^*$ such that $\chi|_{\mathrm{Lie}\, T} = 0$; then $\prod_{w \in W_L} w(\chi)$ is in $H_S^{W_L}$ and nonzero but goes to zero in $H_T^{W_M}$) and $W_L = W_M$ (since we know $S = T$ it is clear that $W_M \subset W_L$; by [Hum90, Prop. 3.6] H_S is a free $H_S^{W_L}$-module of rank $|W_L|$, and H_T is a free $H_T^{W_M}$-module of rank $|W_M|$; since $H_S = H_T$ and $H_S^{W_L} = H_T^{W_M}$ we must have $W_M = W_L$),

and hence $L = M$ (see e. g. [DW98, Thm. 6.2]); this yields (c) since M was an arbitrary maximal compact subgroup of H.

Proposition 3.43 can be applied to $\mathsf{k} = \mathbb{R}$, $K = H_G = H_S^{W_L}$ (which has global dimension s, cf. [Bou07, X.§8.6, Cor. 2]), and the dg K-algebra $A = H_H = H_T^{W_M}$ (note that $H(A) = A$ is a finitely generated K-module since H_S is a finitely generated module over the Noetherian ring K and has A as a subquotient). This shows that (a) and (b) are equivalent.

If $H(G/H) = \mathbb{R}$, then the Leray-Hirsch Theorem can be applied to $EG/H \to EG/G$ and shows that $H_G \to H_H$ is an isomorphism. Hence (d) implies (b).

It remains to show that (c) implies (d). As above let M be a maximal compact subgroup of H. Assume that M is also a maximal compact subgroup of G. Since $H/M \cong \mathbb{R}^h$ and $G/M \cong \mathbb{R}^g$ for suitable $g, h \in \mathbb{N}$, the long exact sequence of homotopy groups of the fiber bundle $G/M \to G/H$ with fiber H/M shows that $\pi_n(G/H) = \{1\}$ for all $n \in \mathbb{N}$. The Hurewicz Theorem then shows that $H(G/H) = \mathbb{R}$. □

Proposition 4.11. *Let G be a connected complex affine algebraic group and $H \subset G$ a closed connected subgroup. Then the equivalent conditions of Proposition 4.10 are equivalent to the following condition:*

(e) $G/H \cong \mathbb{C}^n$ as complex varieties for some $n \in \mathbb{N}$.

Proof. Obviously (e) implies (d). Assume that (c) is satisfied. We claim that the unipotent radical U of G acts transitively on G/H. Let $S \subset G$ be a Levi subgroup, and $L \subset S$ a maximal compact subgroup. Our assumption implies that $gLg^{-1} \subset H$ for some $g \in G$. Since L is Zariski-dense in S this implies that already $S' := gSg^{-1} \subset H$. But then $G = US' = UH$ and hence $U/(U \cap H) \xrightarrow{\sim} G/H$. Now use the (presumably well known) Lemma 4.12. □

We could not find a good reference for the following result.

Lemma 4.12. *Let $V \subset U$ be a closed subgroup of a unipotent complex affine algebraic group. Then $U/V \cong \mathbb{C}^n$ as complex varieties for some $n \in \mathbb{N}$.*

Proof. (We learned this proof from Hanspeter Kraft [Kra11].) We prove this by an outer induction on $\dim(U)$ and an inner induction on $\dim(U/V)$, the cases $\dim(U) = 0$ and $\dim(U/V) = 0$ being trivial. So assume that $\dim(U/V) > 0$.

Let Z be the (nontrivial) center of U. If $Z \subset V$, then $U/V \cong (U/Z)/(V/Z)$ and we can use induction for $V/Z \subset U/Z$. So assume that $Z \subsetneq V$.

Claim: There is a closed subgroup $V \subset V' \subset U$ such that V is normal in V' and $V'/V \cong (\mathbb{C}, +)$.

Let $V'' := VZ$. This is a closed subgroup satisfying $V \subsetneq V'' \subset U$. Note that $Z/(Z \cap V) \cong V''/V$, and these groups are abelian and unipotent. Hence they are isomorphic to a finite product of additive groups $(\mathbb{C}, +)$. Let V' be the inverse image of a one-dimensional subgroup of V''/V. This proves the claim.

The operation of V' on U by right multiplication induces an operation of $V'/V \cong (\mathbb{C}, +)$ on U/V with quotient U/V', and $U/V \to U/V'$ is a principal $(\mathbb{C}, +)$-bundle. Every such bundle over an affine variety is trivial (see [Ser58] or [KS92, Ch. IV]). By induction we know $U/V' \cong \mathbb{C}^m$ for some $m \in \mathbb{N}$. □

Appendix A. Derived fibered base change

We provide a proof of Theorem A.3 in this appendix. This Theorem is a modification of [BL94, p. 56, Lemma C1]; the proof we present is essentially copied from [Soe10].

Sheaves are sheaves of k-modules, for k an arbitrary ring. We denote the category of sheaves on a topological space X by $\mathrm{Sh}(X)$.

Lemma A.1. *Let $f : X \twoheadrightarrow Y$ be an open and surjective morphism of topological spaces (for example a projection or a locally trivial fiber bundle). If all fibers of f are connected, the adjunction morphism $\mathcal{G} \to f_* f^* \mathcal{G}$ is an isomorphism for any sheaf $\mathcal{G}$ on Y.*

Proof. If $U \subset Y$ is open we have to show that $\mathcal{G}(U) \to f^*\mathcal{G}(f^{-1}(U)$ is bijective. It is convenient for this to work with the étale space associated to a sheaf. In this picture we have a pullback diagram

$$f^{-1}(U) \times_U \mathcal{G}|_U \quad = \quad \begin{tikzcd} f^*(\mathcal{G}|_U) \arrow[r, "f'"] \arrow[d, "p'"] & \mathcal{G}|_U \arrow[d, "p"] \\ f^{-1}(U) \arrow[r, "f"] & U \end{tikzcd}$$

and we need to show that continuous sections of p correspond to continuous sections of p' via $t \mapsto (\mathrm{id}_{f^{-1}(U)}, tf)$. This map is obviously injective. If $s = (\mathrm{id}, \tau)$ is a continuous section of p', then $f = p\tau$, and the restriction of τ to any fiber of f is constant. Since Y carries the final topology with respect to $f : X \to Y$, this implies that there is a continuous map $s : U \to \mathcal{G}|_U$ such that $sf = \tau$, and hence $ps = \mathrm{id}_U$. □

Let (P) be a property of topological spaces. We say that a topological space X is **open-locally (P)** if any neighborhood of any point $x \in X$ contains an open neighborhood U of x in X such that U has property (P).

Let

$$
\begin{tikzcd}
W \arrow[r, "g"] \arrow[d, "q"] & V \arrow[d, "p"] \\
Y \arrow[r, "f"] & X
\end{tikzcd}
\tag{A.1}
$$

be a Cartesian diagram of topological spaces.

Theorem A.2 (Fibered base change, cf. [Soe10].). *In the above setting* (A.1) *assume that* p *and* q *are locally trivial fiber bundles with open-locally connected fiber. Then the obvious natural transformation*

$$p^* \circ f_* \to g_* \circ q^*$$

is an isomorphism of functors $\mathrm{Sh}(Y) \to \mathrm{Sh}(V)$.

Proof. We can assume without loss of generality that p is globally trivial. Then (A.1) becomes

$$
\begin{tikzcd}
W = Y \times Z \arrow[r, "g=f\times \mathrm{id}_Z"] \arrow[d, "q"] & V = X \times Z \arrow[d, "p"] \\
Y \arrow[r, "f"] & X,
\end{tikzcd}
\tag{A.2}
$$

where Z is some open-locally connected topological space, and p and q are the first projections. Let $\mathcal{F} \in \mathrm{Sh}(Y)$. It is enough to show that for any point $(x, z) \in X \times Z$ the morphism

$$(p^* f_* \mathcal{F})_{(x,z)} \to (g_* q^* \mathcal{F})_{(x,z)}$$

is an isomorphism. The open neighborhoods of (x, z) in $X \times Z$ have a cofinal subsystem formed by neighborhoods of the form $X' \times Z'$, where X' is an open neighborhood of x in X and Z' is an open connected neighborhood of z in Z. Hence it is enough to show that

$$(p^* f_* \mathcal{F})(X \times Z) \to (g_* q^* \mathcal{F})(X \times Z)$$

is an isomorphism if we assume in addition that Z is connected. In this case Lemma A.1 shows that

$$(p^* f_* \mathcal{F})(X \times Z) = (p_* p^* f_* \mathcal{F})(X) \xleftarrow{\sim} (f_* \mathcal{F})(X) = \mathcal{F}(Y)$$

and

$$(g_* q^* \mathcal{F})(X \times Z) = (q^* \mathcal{F})(Y \times Z) = (q_* q^* \mathcal{F})(Y) \xleftarrow{\sim} \mathcal{F}(Y).$$

Under these identifications the above morphism corresponds to the identity of $\mathcal{F}(Y)$. $\square$

A map $f : X \to Y$ is called **pre-** ∞ **-acyclic** if the adjunction morphism $\mathcal{F} \to Rf_*(f^*(\mathcal{F}))$ is an isomorphism for any any sheaf $\mathcal{F}$ on Y. We say that a topological space is **pre-** ∞ **-acyclic** if $X \to \mathrm{pt}$ is pre- ∞ -acyclic.

Theorem A.3 (Derived fibered base change, cf. [BL94, p. 56, Lemma C1], [Soe10])**.** *In the above setting* (A.1) *assume that* p *and* q *are locally trivial fiber bundles with open-locally pre-*∞*-acyclic fiber. Then the obvious natural transformation*

$$p^* \circ Rf_* \to Rg_* \circ q^*$$

is an isomorphism of functors $D^+(Y) \to D^+(V)$.

Proof. Again we can assume that p is globally trivial, so that (A.1) is given by (A.2) where Z is now an open-locally pre-∞-acyclic topological space.

Step 1: Assume in the situation (A.2) that Y has the discrete topology. If $\mathcal{F}$ is a sheaf on Y, then $q^*(\mathcal{F})$ is g_*-acyclic.

Fix $(x,z) \in X \times Z$. It is enough to show that the stalk $(R^i g_*(q^*(\mathcal{F})))_{(x,z)}$ vanishes for all $i > 0$. We can compute this stalk as the colimit of

$$U \mapsto H^i(g^{-1}(U); q^*(\mathcal{F})),$$

where U ranges over the open neighborhoods of (x,z) in $X \times Z$ of the form $U = X' \times Z'$, where X' is an open neighborhood of $x \in X$ and Z' is an open pre-∞-acyclic neighborhood of z in Z. Fix such a neighborhood $U = X' \times Z'$, and put $Y' := f^{-1}(X')$. Then

$$H^i(g^{-1}(U); q^*(\mathcal{F})) = H^i(Y' \times Z'; q^*(\mathcal{F})) = \prod_{y' \in Y'} H^i(\{y'\} \times Z'; q^*(\mathcal{F})|_{\{y'\} \times Z'})$$

If $c : Z' \to \mathrm{pt}$ is the projection, then, for every $y' \in Y'$, we have

$$H^i(\{y'\} \times Z'; q^*(\mathcal{F})|_{\{y'\} \times Z'}) = H^i(Z'; c^*(\mathcal{F}_{y'})) = H^i(Rc_*(c^*(\mathcal{F}_{y'}))),$$

and this vanishes for $i > 0$ by the assumption on Z'.

Step 2: Let $d : Y' \to Y$ be the identity map, where Y' is the set Y equipped with the discrete topology. We consider the situation (A.2). If $\mathcal{F}$ is a sheaf on Y, then $d_* d^*(\mathcal{F})$ is f_*-acyclic and $q^* d_* d^*(\mathcal{F})$ is g_*-acyclic.

The first claim is obvious since $d_* d^*(\mathcal{F})$ is the flabby sheaf of discontinuous sections of the étale space of $\mathcal{F}$; this is the first step in the Godement resolution of $\mathcal{F}$. To prove the second claim, we expand diagram (A.2) to

$$\text{(A.3)} \qquad \begin{tikzcd} Y' \times Z \arrow[r, "d' := d \times \mathrm{id}_Z"] \arrow[d, "r"] & W = Y \times Z \arrow[r, "g = f \times \mathrm{id}_Z"] \arrow[d, "q"] & V = X \times Z \arrow[d, "p"] \\ Y' \arrow[r, "d"] & Y \arrow[r, "f"] & X \end{tikzcd}$$

where r is the projection and $d' := d \times \mathrm{id}_Z$. Step 1 applied to $d^*(\mathcal{F})$ shows that the sheaf $\mathcal{E} := r^* d^*(\mathcal{F})$ is acyclic both for d'_* and $(g \circ d')_*$. The Leray-Grothendieck spectral sequence then shows that the sheaf $d'_*(\mathcal{E})$ is g_*-acyclic.

Note that any pre-∞-acyclic space is connected. So we can apply Theorem A.2 to the left square in (A.3) and obtain an isomorphism

$$q^* d_* d^*(\mathcal{F}) \xrightarrow{\sim} d'_* r^* d^*(\mathcal{F}) = d'_*(\mathcal{E})$$

of sheaves. Hence $q^* d_* d^*(\mathcal{F})$ is g_*-acyclic.

Step 3: Let $\mathcal{F}$ be a sheaf on Y. Let $\mathcal{F} \hookrightarrow \mathcal{G}$ be its Godement resolution. Since all components of $\mathcal{G}$ are the image under $d_* d^*$ of some sheaf on Y, Step 2 shows that the morphism $p^*(Rf_*(\mathcal{F})) \to Rg_*(q^*(\mathcal{F}))$ is given by $p^*(f_*(\mathcal{G}) \to g_*(q^*(\mathcal{G}))$, and this morphism is an isomorphism by Theorem A.2.

Now, using suitable truncation functors, it is easy to generalize this result from the sheaf $\mathcal{F}$ to arbitrary objects of $D^b(Y)$ and $D^+(Y)$. $\square$

References

[Bek00] Tibor Beke, *Sheafifiable homotopy model categories*, Math. Proc. Cambridge Philos. Soc. **129** (2000), no. 3, 447–475.

[BL94] Joseph Bernstein and Valery Lunts, *Equivariant sheaves and functors*, Lecture Notes in Mathematics, vol. 1578, Springer-Verlag, Berlin, 1994.

[Bor53] Armand Borel, *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207.

[Bor95] ———, *Sous-groupes compacts maximaux des groupes de Lie*, Séminaire Bourbaki, Vol. 1, Soc. Math. France, Paris, 1995, pp. Exp. No. 33, 271–279.

[Bou07] N. Bourbaki, *Éléments de mathématique. Algèbre. Chapitre 10. Algèbre homologique*, Springer-Verlag, Berlin, 2007, Reprint of the 1980 original [Masson, Paris; MR0610795].

[BvdB03] A. Bondal and M. van den Bergh, *Generators and representability of functors in commutative and noncommutative geometry*, Mosc. Math. J. **3** (2003), no. 1, 1–36, 258.

[Dri04] Vladimir Drinfeld, *DG quotients of DG categories*, J. Algebra **272** (2004), no. 2, 643–691.

[DW98] W. G. Dwyer and C. W. Wilkerson, *The elementary geometric structure of compact Lie groups*, Bull. London Math. Soc. **30** (1998), no. 4, 337–364.

[ELO09] Alexander I. Efimov, Valery A. Lunts, and Dmitri O. Orlov, *Deformation theory of objects in homotopy and derived categories. I. General theory*, Adv. Math. **222** (2009), no. 2, 359–401.

[Fre09] Benoit Fresse, *Modules over operads and functors*, Lecture Notes in Mathematics, vol. 1967, Springer-Verlag, Berlin, 2009.

[Hoc65] G. Hochschild, *The structure of Lie groups*, Holden-Day Inc., San Francisco, 1965.

[Hov99a] Mark Hovey, *Model categories*, Mathematical Surveys and Monographs, vol. 63, American Mathematical Society, Providence, RI, 1999.

[Hov99b] Mark Hovey, *Model category structures on chain complexes of sheaves*, arXiv:math/9909024 (1999).

[Hum90] James E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, vol. 29, Cambridge University Press, Cambridge, 1990.

[Kal05] V. Yu. Kaloshin, *A geometric proof of the existence of Whitney stratifications*, Mosc. Math. J. **5** (2005), no. 1, 125–133.

[Kel94] Bernhard Keller, *Deriving DG categories*, Ann. Sci. École Norm. Sup. (4) **27** (1994), no. 1, 63–102.

[Kel05] G. M. Kelly, *Basic concepts of enriched category theory*, Repr. Theory Appl. Categ. (2005), no. 10, vi+137, Reprint of the 1982 original [Cambridge Univ. Press, Cambridge; MR0651714].

[Kel06] Bernhard Keller, *On differential graded categories*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[Kel08] ———, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 467–489.

[Kra11] Hanspeter Kraft, *Email to Olaf Schnürer, 12 Nov. 2011.*

[KS92] Hanspeter Kraft and Gerald W. Schwarz, *Reductive group actions with one-dimensional quotient*, Inst. Hautes Études Sci. Publ. Math. (1992), no. 76, 1–97.

[KS06] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006.

[LC07] Jue Le and Xiao-Wu Chen, *Karoubianness of a triangulated category*, J. Algebra **310** (2007), no. 1, 452–457.

[Lun10] Valery A. Lunts, *Categorical resolution of singularities*, J. Algebra **323** (2010), no. 10, 2977–3003.

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009.

[Nee92] Amnon Neeman, *The connection between the K-theory localization theorem of Thomason, Trobaugh and Yao and the smashing subcategories of Bousfield and Ravenel*, Ann. Sci. École Norm. Sup. (4) **25** (1992), no. 5, 547–566.

[Sch11] Olaf M. Schnürer, *Perfect derived categories of positively graded DG algebras*, Appl. Categ. Structures **19** (2011), no. 5, 757–782.

[Ser58] Jean-Pierre Serre, *Espaces fibrés algébriques*, Séminaire C. Chevalley; 2e année: 1958. Anneaux de Chow et applications, Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1958, pp. 1–37.

[Shk07] Dmytro Shklyarov, *On Serre duality for compact homologically smooth DG algebras*, Preprint (2007), arXiv:math/0702590v1 [math.RA].

[Soe10] Wolfgang Soergel, Werkbank (2010).

[Tab05] Goncalo Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris **340** (2005), no. 1, 15–19.

[Toë09] Bertrand Toën, *Finitude homotopique des dg-algèbres propres et lisses*, Proc. Lond. Math. Soc. (3) **98** (2009), no. 1, 217–240.

[Toë11] ———, *Lectures on DG-categories*, Topics in algebraic and topological K-theory, Lecture Notes in Math., vol. 2008, Springer, Berlin, 2011, pp. 243–302.

[TV07] Bertrand Toën and Michel Vaquié, *Moduli of objects in dg-categories*, Ann. Sci. École Norm. Sup. (4) **40** (2007), no. 3, 387–444.

Department of Mathematics, Indiana University, Rawles Hall, 831 East 3rd Street, Bloomington, IN 47405, USA

E-mail address: vlunts@indiana.edu

Mathematisches Institut, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: olaf.schnuerer@math.uni-bonn.de